A Lion chases down an Antelope

Suppose an antelope is running along a straight path at a constant speed k. Without loss of generality suppose he is running along the x axis.

A lion starts chasing the antelope at a speed 2k in such a way that the lion's direction of motion at any time is along his line of site to the antelope   i.e. the tangent line to the lion's path has as its x -intercept the position of the antelope.

Again without loss of generality assume the lion is on the y axis a distance d when she first sees

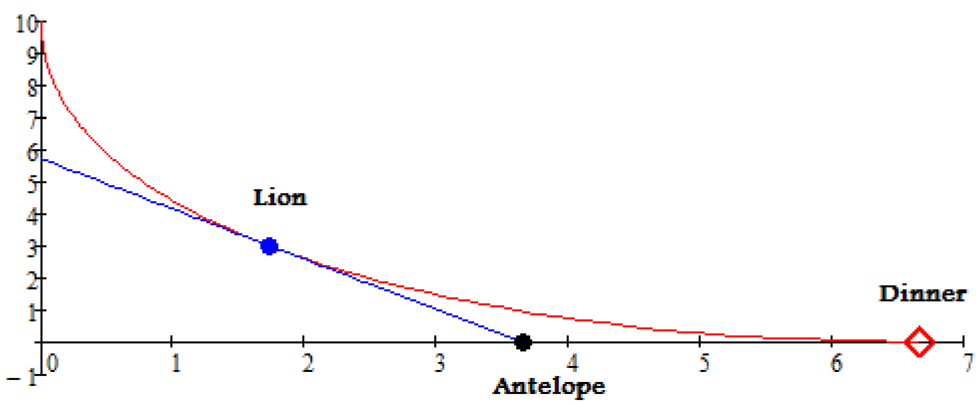
the antelope at the origin.

See the Animation Lion/Antelope and the diagram below
(In the animation we take d = 10units)

1. What is the lion's trajectory

2. Where does the lion catch the antelope?

3. How long does it take to catch the antelope?



The Solution

Let the trajectory of the lion be defined parametrically by:

x = x(t)   and y = y(t)

Then the lion's speed is : $2k = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Squaring both sides we obtain:

$$4k^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

divide by $\left(\frac{dy}{dt}\right)^2$ to obtain:

$$4k^2 \cdot \left(\frac{dt}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1$$

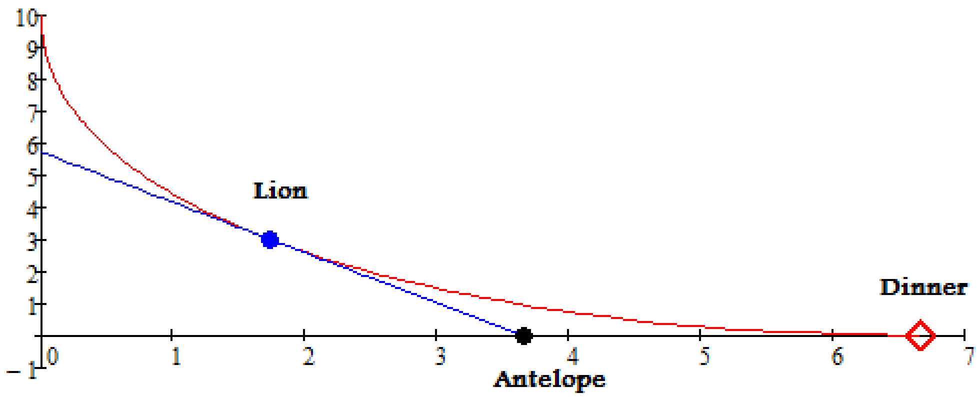
Let q = $\frac{dx}{dy}$

$$4k^2 \cdot \left(\frac{dt}{dy}\right)^2 = q^2 + 1$$

solving for $\frac{dt}{dy}$ we obtain:

$$\frac{dt}{dy} = -\left(\frac{1}{2 \cdot k} \cdot \sqrt{q^2 + 1}\right)$$ We'll refer to this as EQN 1

Now we turn our attention to the tangent line



Let X and Y be the variables describing the tangent line and x(t) and y(t) denote the position of the lion at time t. Then the tangent line at this point in time is:

$$Y = \frac{dy}{dx}(X - x(t)) + y(t)$$

The antelope is running at constant speed k so his position at time t is kt. Therefore the coordinates of the x intercept are (kt,0).

Substituting these coordinates into the equation of the tangent line we obtain:

$$0 = \frac{dy}{dx} \cdot (kt - x(t)) + y(t)$$

Rearranging we obtain :

$$-\frac{dx}{dy} \cdot y = kt - x$$

but recall $\frac{dx}{dy} = q$ therefore we obtain

$$-q \cdot y = kt - x$$

Now differentiate with respect to y to obtain:

$$-q - y \cdot \frac{dq}{dy} = k \cdot \frac{dt}{dy} - \frac{dx}{dy} = k \cdot \frac{dt}{dy} - q$$

Simplifying:

$$-y \cdot \frac{dq}{dy} = k \cdot \frac{dt}{dy} = \frac{-k}{2 \cdot k} \cdot \sqrt{q^2 + 1}$$ (from Eqn 1)

$$y \cdot \frac{dq}{dy} = \frac{1}{2} \cdot \sqrt{q^2 + 1}$$ Eqn 2

To solve Eqn 2 we can simply separate the variables to obtain:

$$2 \frac{dq}{\sqrt{q^2 + 1}} = \frac{dy}{y}$$

Recall $\int \frac{1}{\sqrt{q^2 + 1}} \, dq \rightarrow \operatorname{asinh}(q)$ where $\operatorname{asinh}(q)$ is the inverse hyperbolic sine

We obtain $2 \cdot \operatorname{asinh}(q) = \ln(y) + c$

When y = d then dx/dy i.e. q = 0 therefore c = -ln(d)

$$2 \cdot \operatorname{asinh}(q) = \ln(y) - \ln(d)$$

$$\operatorname{asinh}(q) = \ln\left(\frac{y}{d}\right)^{\frac{1}{2}}$$

$$q = \sinh\left(\ln\left(\frac{y}{d}\right)^{\frac{1}{2}}\right) = \frac{1}{2} \cdot \left(e^{\ln\left(\frac{y}{d}\right)^{\frac{1}{2}}} - e^{-\ln\left(\frac{y}{d}\right)^{\frac{1}{2}}}\right)$$

$$q = \frac{1}{2} \cdot \left(\frac{y}{d}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{d}{y}\right)^{\frac{1}{2}}$$

Recall $q = \frac{dx}{dy}$

$$\frac{dx}{dy} = \frac{1}{2} \cdot \left(\frac{y}{d}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{d}{y}\right)^{\frac{1}{2}}$$

Separating the variables and doing the rather simple integration we obtain:

$$x(y) = y^{\frac{1}{2}} \cdot d^{\frac{1}{2}} - \frac{1}{3}\left(\frac{y^3}{d}\right)^{\frac{1}{2}} + c$$

When x = 0 then y = d therefore c = 2d/3

Finally we obtain $x(y) = y^{\frac{1}{2}} \cdot d^{\frac{1}{2}} - \frac{1}{3}\left(\frac{y^3}{d}\right)^{\frac{1}{2}} + 2\frac{d}{3}$

It follows the lion catches the antelope when y = 0 and x = 2d/3

since the speed of the antelope is k the time is $kt = 2\frac{d}{3}$ or at $t = \frac{2d}{3k}$.

For example suppose the lion is 600 ft away initially and the antelope runs at 22ft/sec

Then the lion catches the antelope 400 feet down range and it takes 18 secs.

Note We could also use this same analysis for say using a heat seeking missile to blow up a plane.