Work

Suppose an object is moving in one dimension either horizontally or vertically. Suppose a Force which is constant in magnitude and in the same direction as the object's motion acts on that object.

We define the work done on that object as W = Force x Distance.

For example a 10 kg object is raised by a rope on a pulley a distance of 15 meters. Suppose for now we neglect the weight of the rope. (We'll add this in later)

See Animation 1.

The force of gravity is $F = mg = 10\cdot(-9.8) = -98N$ We must exert a force of 98N to raise the object.

The work done against gravity is $F\cdot d = 98N\cdot 15m = 1470Nm$. One Newton- meter is called a Joule and the Work done is 1470 Joules.

In the English system the unit of work is the ft-lb. In the cgs system one dyne-cm is called an erg.

But what is Work exactly?

The work done is the change in the sum of potential and kinetic energy: $W = \Delta E_p + \Delta E_k$.

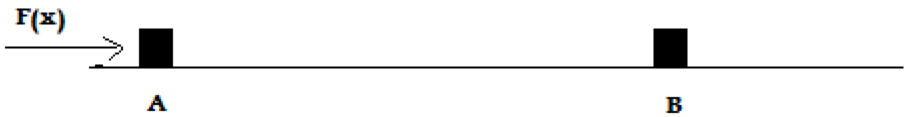
For our example the object is at rest initially and is at rest when we stop. Therefore the 1470 Joules is the change in the potential energy of the body.

If we allow the object to drop all the potential energy is converted to kinetic energy when it hits the ground and we can exploit this to calculate the velocity of the object when it hits the ground. Recall $E_k = \frac{1}{2}\cdot m\cdot v^2$

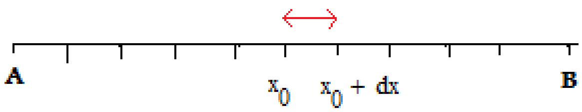
therefore $\frac{1}{2}\cdot m\cdot v^2 = 147($ from which we obtain $v = 17.1\frac{m}{s}$ when the object hits the ground.

So what does all of this have to do with Calculus? Until we get to vector calculus and discuss Line Integrals we are somewhat restricted in what we can do as we are restricted to one dimensional motion. However we can consider a class of problems in which we can at least drop the restriction that F be constant in magnitude such as with the motion of a mass on a spring . This will lead us to Work as an Integral.

Suppose a force F = F(x) acts on an object as the object moves from pt A to pt B .



We partition the path into a large number of segments of length dx.



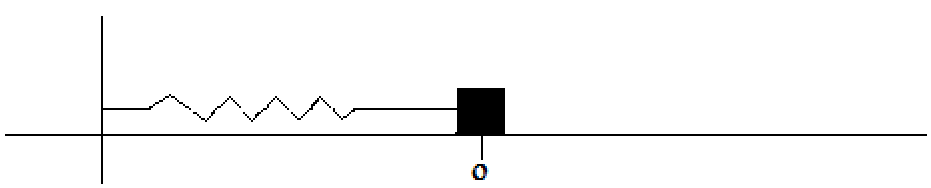
If F(x) is continuous then over each segment from x_0 to $x_0 + dx$ F(x) is approximately constant and equal to $F(x_0)$. The differential amount of work done on this subinterval is then

$dW = F(x_0) \cdot dx$. (Formally we could use Δ x and take appropriate limits)

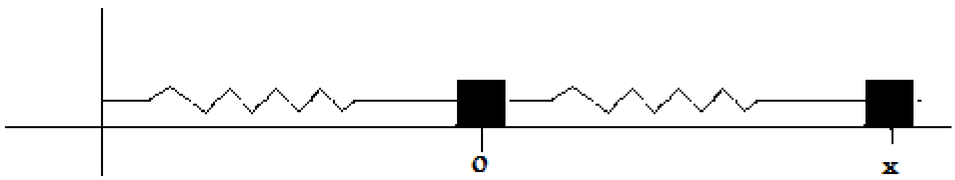
It follows then the total work is the continuous sum i.e. the integral $W = \int_A^B F(x)\, dx$.

Example 1

Spring problems and Hooke's Law. Suppose a mass is connected to a spring and is at rest.



If the spring is stretched (or compressed) so the object is at point x Hooke's Law states the force the spring exerts on the object is $F(x) = -k \cdot x$ where k is a constant determined empirically and depends on the physical properties of the spring. Why the negative sign? if x > 0 the restoring force is to the left and it x < 0 the restoring force is to the right. In order to stretch the spring we would need to exert a force of $F(x) = k \cdot x$



Suppose it takes 7N to stretch a spring 0.5 meters.

a. How much work was done?

b.How much work is required to stretch it and additional .5 m ?

a. First we must determine k.

$F(x) = k \cdot x$

$7 = k \cdot .5$

$k = 14$ and $F(x) = 14x$

$$W = \int_0^{.5} 14x\,dx = 1.75J$$

b. $W = \int_{.5}^{1} 14x\,dx = 5.25J$ This is not surprising as the further the spring is stretched more Force is required.

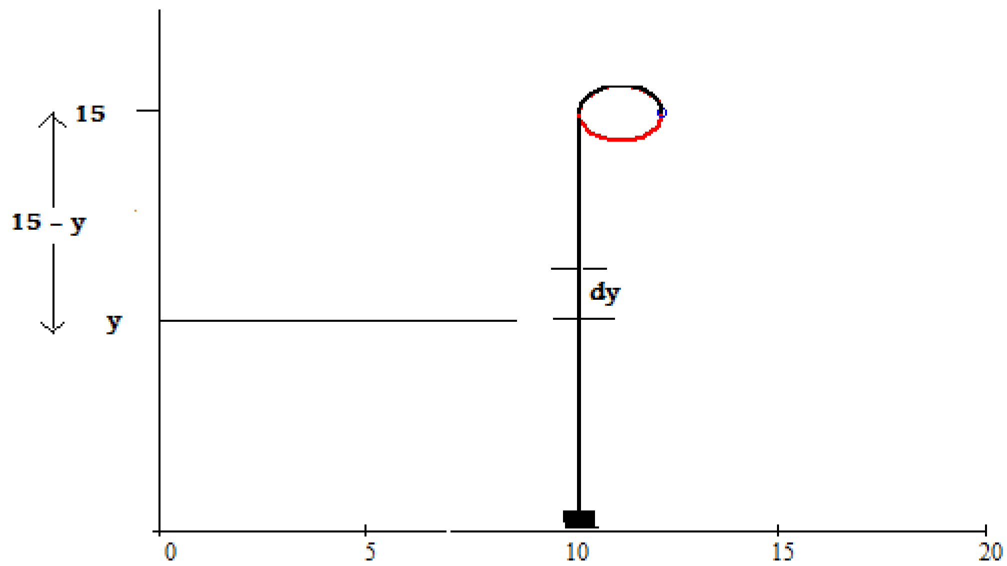
Example 2

Let's return to our pulley problem and this time we'll factor in the weight of the rope.

A 10 kg object is raised by a rope on a pulley a distance of 15 meters. The work done on the object attached to the rope is still 1470 J.

Suppose the linear density of the rope is $\lambda = 0.1$kg/m.

Suppose we divide the rope into a large number of segments of length dy.



For each segment the weight is λ dy = .1dy . The distance traveled is 15 - y .

For an infinitesimal segment dW = .1(15-y)dy.

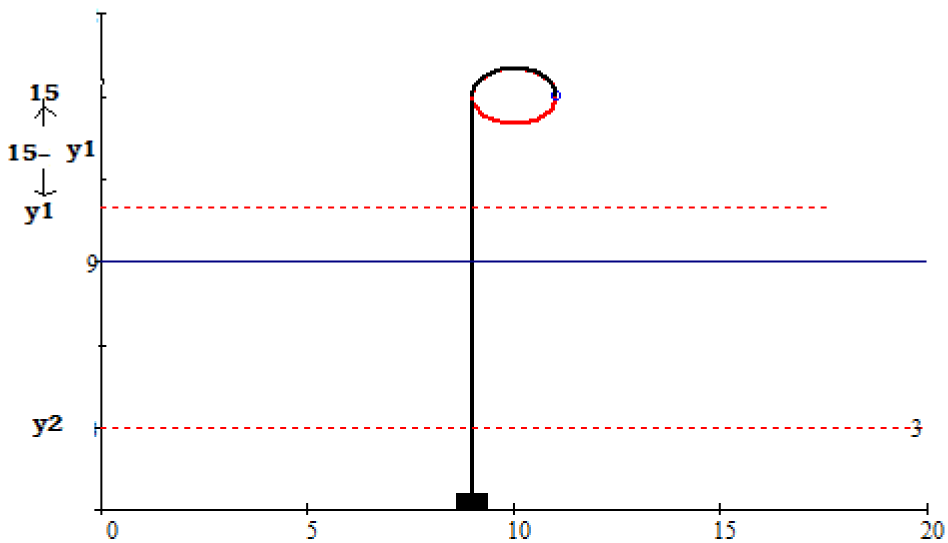
Then the total work is $W = \int_0^{15} .1\cdot(15-y)\,dy = 11.25\mathrm{J}$

The total work is then $11.25 + 1470 = 1481.25\mathrm{J}$ Recall initially we said assume the weight of the rope is negligible. In this case this is not a bad assumption -less than 1% of the total work. But what if we had a chain with $\lambda = 2$kg/m ?

$W = \int_0^{15} 2\cdot(15-y)\,dy = 225\mathrm{J}$ which is not negligible -- 13.3% of the total work.

Suppose we only wind 6m of the rope? See Animation 2.

Here the complication is that every segment below the 9m mark travels 6m but above the 9m mark each segment travels 15 - y meters



We simply use 2 integrals $W = \int_0^9 .1\cdot(6)\,dy + \int_9^{15} .1\cdot(15 - y)\,dy = 5.4J + 1.8J = 7.2J$

The total work is $W = 98\cdot 6 + 7.2 = 595.$ (the object is raised only 6m also)

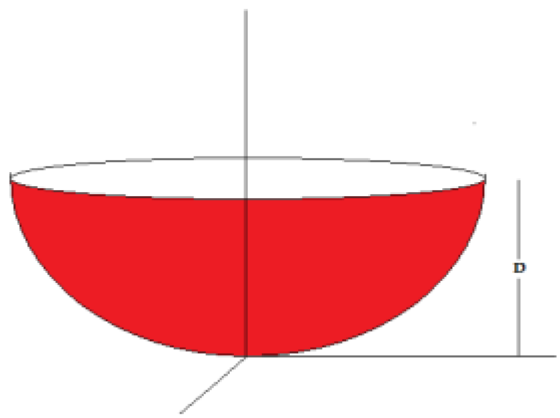
Suppose now the object we are raising is a bucket but the bucket is leaking so that its mass is 10kg on the ground but leaks at a constant rate so that its mass is 5kg when it reaches the 9m mark ?

$$\text{mass} = 10 - \frac{5}{9}\cdot y = 10 - .556y$$

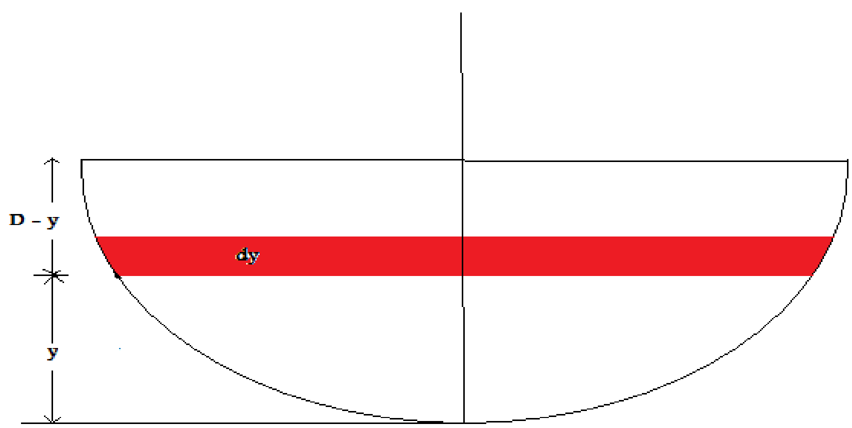
$$W = \int_0^9 10 - .556y\,dy + 7.2 = 67.5J + 7.2J = 74.7J$$

Example 3 Work Required to Empty a Tank

Suppose a tank is filled with a fluid of density ρ . How much work is required to pump all the fluid to the top?



We partition the tank into cross-sections and calculate the mass of each cross-section and calculate the work done to raise each cross-section to the top.



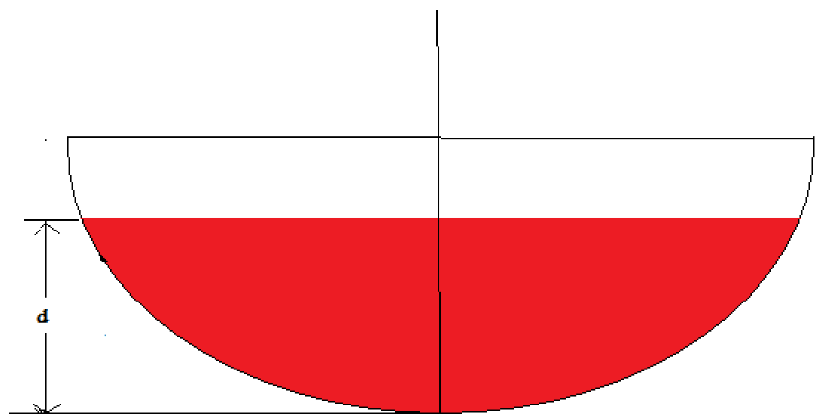
The volume of this cross-section is V = A(y)dy where A is the cross-sectional area.

Since $\rho = \frac{m}{V}$ it follows m = ρ V = ρ A(y)dy. The weight is then mg = ρ gA(y)dy . The distance this cross-section is pumped is D - y.

Then dW = m g (D-y) = ρ g A(y) (D-y) dy

$W = \int_0^D \rho \cdot g \cdot A(y) \cdot (D - y)\,dy$. If ρ is a constant then $W = \rho \cdot g \cdot \int_0^D A(y) \cdot (D - y)\,dy$

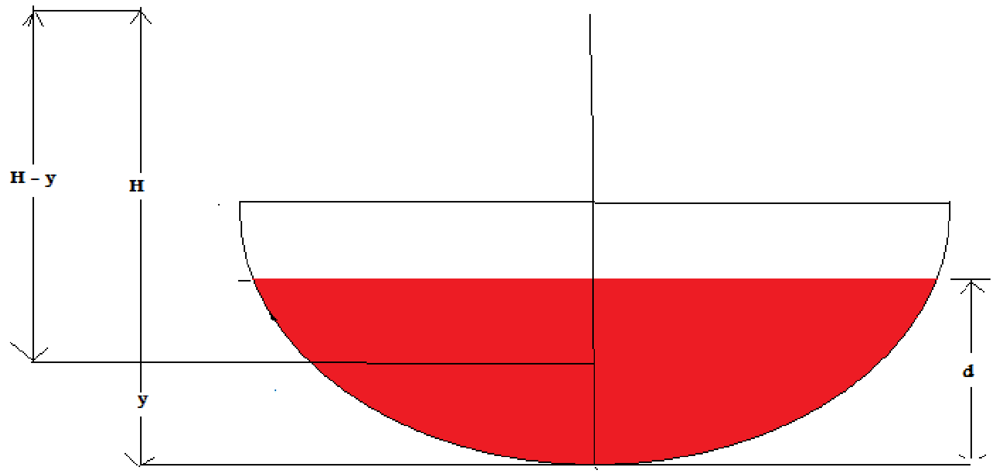
Suppose the tank is only filled to a depth d. How does the Integral change?



It affects only the integration limits i.e. we only add up the cross-sections from 0 to d.

$$W = \rho \cdot g \cdot \int_0^d A(y) \cdot (D - y)\,dy.$$

What if the tank is filled to a depth d but we pump the fluid to a height H ? For example an underground tank where the fluid is pumped to a height 2m above the ground.



Now each cross-section is pumped a distance H - y .

$$W = \rho \cdot g \cdot \int_0^d A(y) \cdot (H - y)\, dy$$

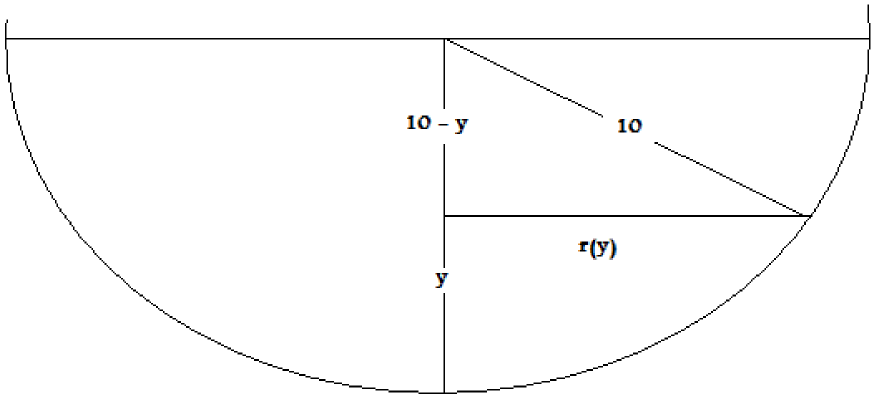
Note that in any problem the only effort is in calculating the cross-sectional area as a function of y.

Example Suppose we have a hemispherical tank of radius 10m. It is filled to a depth of 3m.

The density of the fluid is 5kg/m^3.

Calculates the work required to pump the fluid to a height of 12m.

$$W = 5 \cdot (9.8) \cdot \int_0^3 A(y) \cdot (12 - y)\, dy$$



The cross-sections are circles - $A(y) = \pi \cdot r(y)^2$

To get r(y) we use the right triangle above to obtain $A(y) = \pi \cdot \left[100 - (10 - y)^2\right] = \pi \cdot \left(20 \cdot y - y^2\right)$

$$W = 49 \cdot \pi \cdot \int_0^3 \left(20 \cdot y - y^2\right) \cdot (12 - y)\, dy = \frac{159201 \cdot \pi}{4} = 1.25 \times 10^5 J$$