Show that if F is a force field with constant magnitude k pointing outward from the origin

then as a particle travels along the smooth curve y = f(x) as x varies from a to b

is k $((b^2 + f(b)^2)^{1/2} - (a^2 + f(a)^2)^{1/2})$

We start by obtaining an expression for the Force F

For a radial Force we would normally use $\vec{F} = \left(x \cdot \vec{i} + y \cdot \vec{j}\right)$ but since we want a constant magnitude of k we use :

$$\vec{F} = k \cdot \left(\frac{x \cdot \vec{i} + y \cdot \vec{j}}{\sqrt{x^2 + y^2}} \right) = k \cdot \left(\frac{x \cdot \vec{i} + f(x) \cdot \vec{j}}{\sqrt{x^2 + f(x)^2}} \right)$$

Using x as the parameter

$$\vec{r} = x \cdot \vec{i} + f(x) \cdot \vec{j}$$

$$\frac{d\vec{r}}{dx} = \vec{i} + \frac{df}{dx} \cdot \vec{j}$$

Recall $\vec{T}$ is the unit tangent so we use $\dfrac{\frac{d\vec{r}}{dx}}{\left|\frac{d\vec{r}}{dx}\right|}$

$$\vec{T} = \frac{1}{\sqrt{1 + \left(\frac{df}{dx}\right)^2}} \cdot \left(\vec{i} + \frac{df}{dx} \cdot \vec{j} \right)$$

$$\vec{F} \cdot \vec{T} = \frac{1}{\sqrt{1 + \left(\frac{df}{dx}\right)^2}} \cdot \frac{\left(x + \frac{df}{dx} \cdot f \right)}{\sqrt{x^2 + f(x)^2}}$$

Usually when we use t as a parameter $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$

But since x is the parameter

$$ds = \sqrt{1 + \left(\frac{df}{dx}\right)^2} \cdot dx$$

$$\int_a^b F \cdot T\, ds = k \cdot \int_a^b \frac{1}{\sqrt{1 + \left(\frac{df}{dx}\right)^2}} \cdot \frac{\left(x + \frac{df}{dx} \cdot f\right)}{\sqrt{x^2 + f(x)^2}} \cdot \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

$$\int_a^b F \cdot T\, ds = \int_a^b 1 \cdot \frac{\left(x + \frac{df}{dx} \cdot f\right)}{\sqrt{x^2 + f(x)^2}}\, dx$$

Let $u = \sqrt{x^2 + f(x)^2}$ $\qquad du = 1 \cdot \frac{\left(x + \frac{df}{dx} \cdot f\right)}{\sqrt{x^2 + f(x)^2}}$

$$\int \frac{\left(x + \frac{df}{dx} \cdot f\right)}{\sqrt{x^2 + f(x)^2}}\, dx = \int 1\, du = u = \sqrt{x^2 + f(x)^2}$$

$$\int_a^b F \cdot T\, ds = k \cdot \left[\sqrt{x^2 + f(x)^2} \cdot \right]_a^b = k \cdot \left(\sqrt{b^2 + f(b)^2} - \sqrt{a^2 + f(a)^2}\right)$$