Vector Flow Fields

We have seen with our study of vector valued functions given $\vec{r}(t) := x(t)\cdot\vec{i} + y(t)\cdot\vec{j}$ that the velocity is then given by

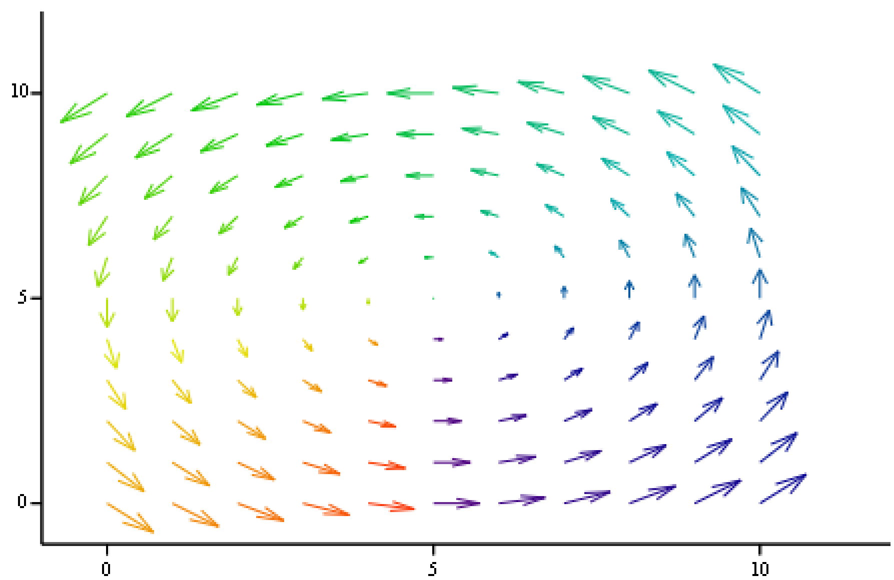
$\vec{v}(t) := \frac{dx}{dt}\cdot\vec{i} + \frac{dy}{dt}\cdot\vec{j}$. with a flow field we start with $\vec{v}(t)$ and at each point in the plane we associate a vector which would give us the velocity of a particle at that point. The graph of all such vectors is called a flow field.

Below is the Flow Field for $\vec{v} := -y\cdot\vec{i} + x\cdot\vec{j}$ where the components are expressed in terms of x and y

$a := -2($ $\quad b := 2($ $\quad c := -2($ $\quad d := 2($ $\quad \Delta x := 4$ $\quad \Delta y := 4$

$i := 0..\frac{b-a}{\Delta x}$ $\qquad j := 0..\frac{(d-c)}{\Delta y}$ $\qquad x_i := a + i\cdot\Delta x$ $\qquad y_j := c + j\cdot\Delta y$

$x(x,y) := -y$ $\qquad y'(x,y) := x$ $\qquad X_{i,j} := x(x_i, y_j)$ $\qquad Y_{i,j} := y'(x_i, y_j)$



(X, Y)

There is an intimate relationship between the flow field and the flow lines. The Flow lines are curves along which a particle would flow unless acted on by an outside force . We generate a differential equation from

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$ and solve this for y.

In our above example we obtain $\frac{dy}{dx} = \frac{-x}{y}$

Separating the variables $y \cdot dy = -x \cdot dx$

Integrating $x^2 + y^2 = c.$

How do we get the flow lines in terms of t ?

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x$$

$$\frac{d^2x}{dt^2} = \frac{-dy}{dt} = -x$$

Which is the simple 2d order equation $\frac{d^2x}{dt^2} = -x$

$\frac{d^2x}{dt^2} + x = 0$ with characteristic equation $\lambda^2 + 1 = 0$ with solutions $\lambda = \pm i$

which yields the solution x(t) = Acos(t) +Bsin(t)

if x(0) = 1 x'(0) = 0

We obtain x(t) = c cos(t)

By the same process we obtain $\dfrac{d^2y}{dt^2} = -y$ with solution y(t) = Acos(t) +Bsin(t)

if y(0) = 0   y'(0) = c

We obtain y(t) = csin(t)

$\overrightarrow{r(t)} = c\cdot\cos(t)\cdot\vec{i} + c\sin(t)\cdot\vec{j}$

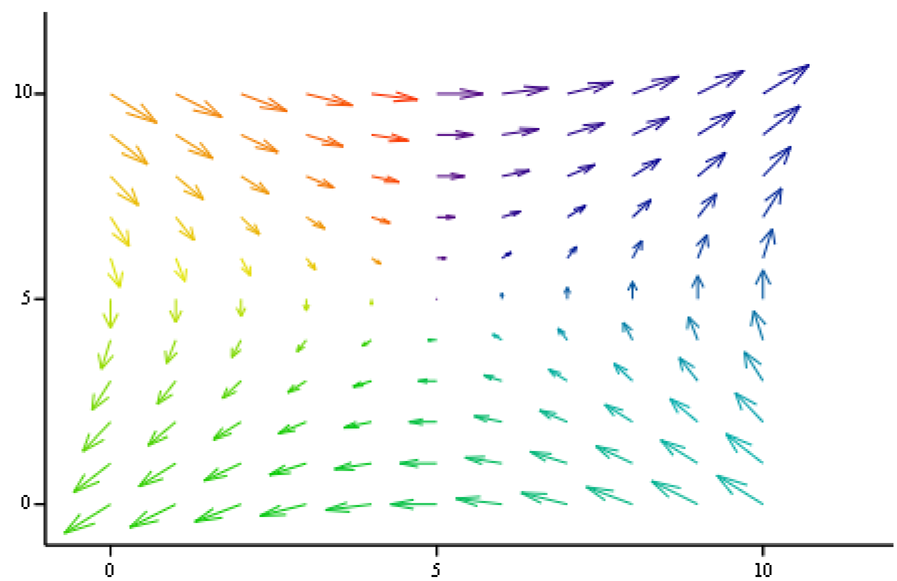
From which we have $\overrightarrow{v(t)} = -c\cdot\sin(t)\cdot\vec{i} + c\cdot\cos(t)\cdot\vec{j} = -y\cdot\vec{i} + x\cdot\vec{j}$ Which is what we should have

Example 2

$a := -20$ $b := 20$ $c := -20$ $d := 20$ $\Delta x := 4$ $\Delta y := 4$

$i := 0..\dfrac{b-a}{\Delta x}$ $j := 0..\dfrac{(d-c)}{\Delta y}$ $x_i := a + i\cdot\Delta x$ $y_j := c + j\cdot\Delta y$

$x'(x,y) := y$ $y'(x,y) := x$ $X_{i,j} := x'\left(x_i, y_j\right)$ $Y_{i,j} := y'\left(x_i, y_j\right)$



(X, Y)

$$\frac{dy}{dx} = \frac{x}{y}$$

Again separating variables and integrating we obtain $\frac{y^2}{c^2} - \frac{x^2}{c^2} = 1$ which are the equations of hyperbolas

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = x$$

$$\frac{d^2 \cdot x}{dt^2} = x \qquad \frac{d^2 \cdot y}{dt^2} = y$$

$\frac{d^2 \cdot x}{dt^2} = x$ which yields $\frac{d^2 \cdot x}{dt^2} - x = 0$ with characteristic equation $\lambda^2 - 1 = 0$ with solutions $\lambda = \pm 1$

$x = Ae^{t} + Be^{-t}$

If x(0) = C and x ' (0) = 0 We obtain $x(t) = C\cosh(t)$

similarly we obtain $y(t) = C \cdot \sinh(t)$

The parametric equations of a hyperbola

$$\vec{r(t)} = C\cosh(t) \cdot \vec{i} + C \cdot \sinh(t) \cdot \vec{j}$$

$$\vec{v(t)} = C\sinh(t) \cdot \vec{i} + C \cdot \cosh(t) \cdot \vec{j} = y \cdot \vec{i} + x \cdot \vec{j}$$

Closely related to the idea of a velocity flow field is the velocity potential--curves which are orthogonal to the flow lines and represent curves where $\overrightarrow{v(t)}$ is the same at every pt. See the lecture Orthognal trajectories for a complete discussion.

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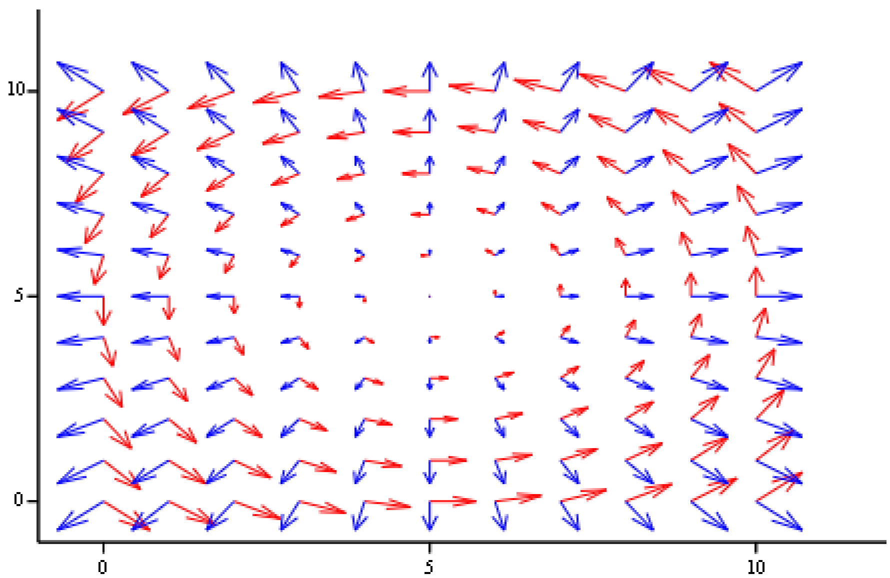
$x'(x,y) := -y$ $\qquad y'(x,y) := x$ $\qquad X_{i,j} := x'(x_i, y_j)$ $\qquad Y_{i,j} := y'(x_i, y_j)$

$x1(x,y) := x$ $\qquad y1'(x,y) := y$ $\qquad X1_{i,j} := x1(x_i, y_j)$ $\qquad Y1_{i,j} := y1'(x_i, y_j)$



(X, Y), (X1, Y1)