# Variation of Parameters

The general solution to a second order non- homogeneous differential equation can be reduced to solving 2 first order differential equations. This differs from reduction of order in that we have 2 independent 1st order equations as opposed to having to solve a first order eqn and then having to integrate that result. Recall that the homogeneous solution is A $y_1(x)$ + B $y_2(x)$ where A and B are constants. With variation of parameters we replace the constants A and B with functions z1(x) and z2(x) and force the result to solve the non-homogeneous DE. We start with :

$$a_2 \cdot \frac{d^2y}{dx^2} + a_1 \cdot \frac{dy}{dx} + a_0 = f(x)$$

Suppose $y_1$ and $y_2$ are homogeneous solutions.

The general solution is then given by:

$$y = y_1 \cdot z_1 + y_2 \cdot z_2$$

Where $z_1$ and $z_2$ are functions yet to be determined.

$$\frac{dy}{dx} = \frac{dy_1}{dx} \cdot z1 + y_1 \cdot \frac{dz_1}{dx} + \frac{dy_2}{dx} \cdot z2 + y_2 \cdot \frac{dz_2}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d^2y_1}{dx^2} \cdot z_1 + 2 \cdot \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} + y_1 \frac{d^2z_1}{dx^2} + \left( \frac{d^2y_2}{dx^2} z_2 + 2 \cdot \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} + y_2 \frac{d^2z_2}{dx^2} \right)$$

Now we plug these results back into the original DE. The first step is to combine all terms multiplying $z_1$ and $z_2$ and recall y1 and y2 are homogeneous solutions :

$$z_1 \cdot \left[ a_2 \cdot \left( \frac{d^2y_1}{dx^2} + a_1 \cdot \frac{dy_1}{dx} + a_0 \cdot y_1 \right) \right] + z_2 \cdot \left[ a_2 \cdot \left( \frac{d^2y_2}{dx^2} + a_1 \cdot \frac{dy_2}{dx} + a_0 \cdot y_2 \right) \right] = 0$$

We can eliminate the following terms:

$$y = \cancel{y_1 \cdot z_1 + y_2 \cdot z_2}$$

$$\frac{dy}{dx} = \cancel{\frac{dy_1}{dx} \cdot z1} + y_1 \cdot \frac{dz_1}{dx} + \cancel{\frac{dy_2}{dx} \cdot z2} + y_2 \cdot \frac{dz_2}{dx}$$

$$\frac{d^2y}{dx^2} = \cancel{\frac{d^2y_1}{dx^2} \cdot z_1} + 2 \cdot \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} + y_1 \frac{d^2 z_1}{dx^2} + \left( \cancel{\frac{d^2y_2}{dx^2} z_2} + 2 \cdot \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} + y_2 \frac{d^2 z_2}{dx^2} \right)$$

We are left with :

$$a_2 \cdot \left( y_1 \frac{d^2 z_1}{dx^2} \right) + a_2 \cdot \left( 2 \cdot \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + a_1 \cdot \left( y_1 \cdot \frac{dz_1}{dx} \right) + \left[ a_2 \cdot \left( y_2 \frac{d^2 z_2}{dx^2} \right) + a_2 \cdot \left( 2 \cdot \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) + a_1 \cdot \left( y_2 \cdot \frac{dz_2}{dx} \right) \right] = f(x)$$

Note :

$$a_2 \cdot \left( y_1 \frac{d^2 z_1}{dx^2} \right) + a_2 \cdot \left( \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + a_2 \cdot \left( y_2 \frac{d^2 z_2}{dx^2} \right) + a_2 \cdot \left( 2 \cdot \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) = a_2 \cdot \frac{d\left( y_1 \frac{dz_1}{dx} + y_2 \cdot \frac{dz_2}{dx} \right)}{dx}$$

and the remaining terms give :

$$a_2 \cdot \left[ \left( \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + a_1 \cdot \left( y_1 \cdot \frac{dz_1}{dx} \right) \right] + a_2 \cdot \left( \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) + a_1 \cdot \left( y_2 \cdot \frac{dz_2}{dx} \right) = a_2 \cdot \left[ \left( \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + \left( \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) \right] + a_1 \cdot \left( y_1 \frac{dz_1}{dx} + y_2 \cdot \frac{dz_2}{dx} \right)$$

We impose the condition $\quad y_1 \frac{dz_1}{dx} + y_2 \cdot \frac{dz_2}{dx} = 0$

Which means then :

$$a_2 \cdot \left[ \left( \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + \left( \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) \right] = f(x)$$

We are left with the system of equations (linear in $\frac{dz_1}{dx}$ and $\frac{dz_2}{dx}$ ) :

$$y_1 \frac{dz_1}{dx} + y_2 \cdot \frac{dz_2}{dx} = 0$$

$$\left( \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + \left( \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) = \frac{f(x)}{a_2}$$

Recall Cramers Rule :

Given ax + by = c

dx + ey = f

Then:

$$x = \frac{\begin{pmatrix} c & b \\ f & e \end{pmatrix}}{\begin{pmatrix} a & b \\ d & e \end{pmatrix}} \qquad y = \frac{\begin{pmatrix} a & c \\ d & f \end{pmatrix}}{\begin{pmatrix} a & b \\ d & e \end{pmatrix}}$$

Applying Cramer's Rule to

$$y_1 \frac{dz_1}{dx} + y_2 \cdot \frac{dz_2}{dx} = 0$$

$$\left( \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + \left( \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) = \frac{f(x)}{a_2}$$

We obtain:

$$\frac{dz_1}{dx} = \frac{\begin{pmatrix} 0 & y_2 \\ \frac{f(x)}{a_2} & \frac{dy_2}{dx} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}}$$

But $\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}$ is the Wronskian W(y1,y2)

$$\frac{dz_1}{dx} = \frac{\begin{pmatrix} 0 & y_2 \\ \frac{f(x)}{a_2} & \frac{dy_2}{dx} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}} = \frac{-y_2 \cdot f(x)}{a_2 \cdot W(y1, y_2)}$$

similarly

$$\frac{dz_2}{dx} = \frac{\begin{pmatrix} y_1 & 0 \\ \frac{dy_1}{dx} & \frac{f(x)}{a_2} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}} = \frac{y_1 \cdot f(x)}{a_2 \cdot W(y1, y_2)}$$

Summary

Given :

$$a_2 \cdot \frac{d^2 y}{dx^2} + a_1 \cdot \frac{dy}{dx} + a_0 = f(x)$$

where $y_1$ and $y_2$ are homogeneous solutions.

The general solution is then given by:

$$y = y_1 \cdot z_1 + y_2 \cdot z_2$$

Where $z_1$ and $z_2$ are given by :

$$\frac{dz_1}{dx} = \frac{\begin{pmatrix} 0 & y_2 \\ \frac{f(x)}{a_2} & \frac{dy_2}{dx} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}} = \frac{-y_2 \cdot f(x)}{a_2 \cdot W(y1, y_2)}$$

$$\frac{dz_2}{dx} = \frac{\begin{pmatrix} y_1 & 0 \\ \frac{dy_1}{dx} & \frac{f(x)}{a_2} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}} = \frac{y_1 \cdot f(x)}{a_2 \cdot W(y1, y_2)}$$

We can use the above equations to solve for $z_1$ and $z_2$ and plug into $y = y_1 \cdot z_1 + y_2 \cdot z_2$ or we can write the general solution as:

$$y = y_1 \cdot \int \frac{-y_2 \cdot f(x)}{a_2 \cdot W(y1, y_2)} \, dx + y_2 \cdot \int \frac{y_1 \cdot f(x)}{a_2 \cdot W(y1, y_2)} \, dx$$

Let's consider a simple example and then we will consider one more challenging

$$\frac{d^2 y}{dx^2} + 2 \cdot y + y = e^{-x} \cdot \sin(x)$$

The homogeneous solutions are :

$$y_1(x) = e^{-x} \qquad y_2(x) = x\cdot e^{-x}$$

$$W\left(y_1, y_2\right) = \begin{pmatrix} e^{-x} & x\cdot e^{-x} \\ -e^{-x} & e^{-x} - x\cdot e^{-x} \end{pmatrix} = e^{-2\cdot x}$$

$$y = y_1\cdot\int \frac{-y_2\cdot f(x)}{a_2\cdot W\left(y1, y_2\right)}\,dx + y_2\cdot\int \frac{y_1\cdot f(x)}{a_2\cdot W\left(y1, y_2\right)}\,dx$$

$$y = e^{-x}\cdot\int \frac{-x\cdot e^{-x}\cdot e^{-x}\cdot \sin(x)}{e^{-2\cdot x}}\,dx + x\cdot e^{-x}\cdot\int \frac{e^{-x}\cdot e^{-x}\cdot \sin(x)}{e^{-2\cdot x}}\,dx$$

$$y = e^{-x}\cdot\int -x\cdot\sin(x)\,dx + x\cdot e^{-x}\cdot\int \sin(x)\,dx$$

$$y = e^{-x}\cdot\left(x\cdot\cos(x) - \sin(x) + c_1\right) + x\cdot e^{-x}\cdot(-\cos(x) + c2)$$

$$y = c_1\cdot e^{-x} + c_2\cdot x\cdot e^{-x} - e^{-x}\cdot\sin(x)$$

Note $c_1\cdot e^{-x} + c_2\cdot x\cdot e^{-x}$ is the homogeneous solution and $-\left(e^{-x}\cdot\sin(x)\right)$ is the particular solution

Example 2

$$\left(D^2 - 2D + 1\right)\cdot y = \frac{e^{2x}}{\left(e^x + 1\right)^2}$$

the homogeneous solutions are

$y1 = e^{x}$ $y2 = x \cdot e^{x}$

$W(y1, y2) = e^{2x}$

The general solution is

$y = e^{x} \cdot z1 + x \cdot e^{x} \cdot z2$

$$\frac{dz1}{dx} = \frac{-x \cdot e^{x} \cdot \frac{e^{2x}}{\left(e^{x} + 1\right)^{2}}}{e^{2x}} = \frac{-x \cdot e^{x}}{\left(e^{x} + 1\right)^{2}}$$

$$\frac{dz2}{dx} = \frac{e^{x} \cdot \frac{e^{2x}}{\left(e^{x} + 1\right)^{2}}}{e^{2x}} = \frac{e^{x}}{\left(e^{x} + 1\right)^{2}}$$

It is fairly easy to show $z2 = \frac{-1}{e^{x} + 1} + c2$

$$z1 = \int \frac{-x \cdot e^{x}}{\left(e^{x} + 1\right)^{2}} \, dx$$

$u = -x \qquad dv = \frac{e^x}{\left(e^x + 1\right)^2}$

$du = -dx$

$v = \frac{-1}{e^x + 1}$

$$\int \frac{-x \cdot e^x}{\left(e^x + 1\right)^2}\,dx = \frac{x}{e^x + 1} - \int \frac{1}{e^x + 1}\,dx$$

Let $u = e^x$

$$\int \frac{1}{e^x + 1}\,dx = \int \frac{e^x}{e^x \cdot \left(e^x + 1\right)}\,dx = \int \frac{1}{u(u + 1)}\,du = -\ln\left(\frac{1 + u}{u}\right) = -\ln\left(1 + e^x\right) + x$$

$$z1 = \frac{x}{e^x + 1} + \ln\left(1 + e^x\right) - x + c1$$

$$y = e^x \cdot z1 + x \cdot e^x \cdot z2$$

$$y = e^x \cdot \left(\frac{x}{e^x + 1} + \ln\left(1 + e^x\right) - x + c1\right) + x \cdot e^x \cdot \left(\frac{-1}{e^x + 1} + c2\right)$$

$$y = c1 \cdot e^x + c2 \cdot x \cdot e^x + e^x \cdot \left(\ln\left(1 + e^x\right) - x\right) = c1 \cdot e^x + c2 \cdot x \cdot e^x + e^x \cdot \ln\left(1 + e^x\right)$$