# Triple Integrals in Rectangular Coordinates

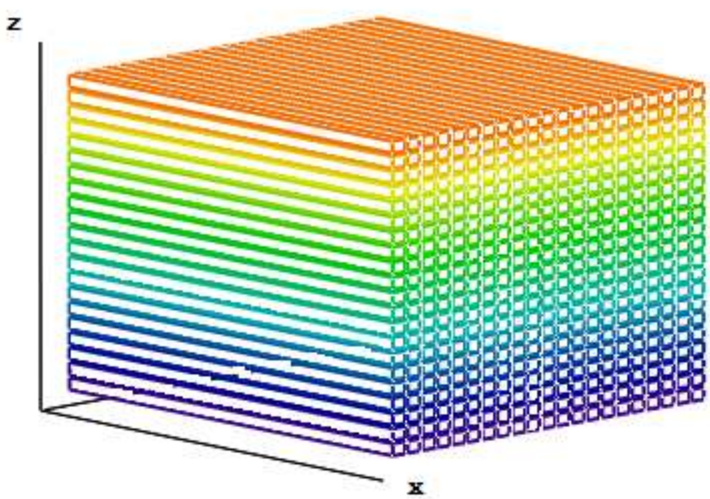
We saw that if f(x,y) is the density of a plate occupying a region in the x-y plane Then $\iint f(x,y)dA$ is the mass of the plate .

Suppose now we have a solid in 3space and f(x,y,z) represents the density at any point in that solid. How do we now calculate the mass ? The first difficulty we run into is that we need 3-dimensions just to graph the domain. The density function will not appear in any graph or animation.

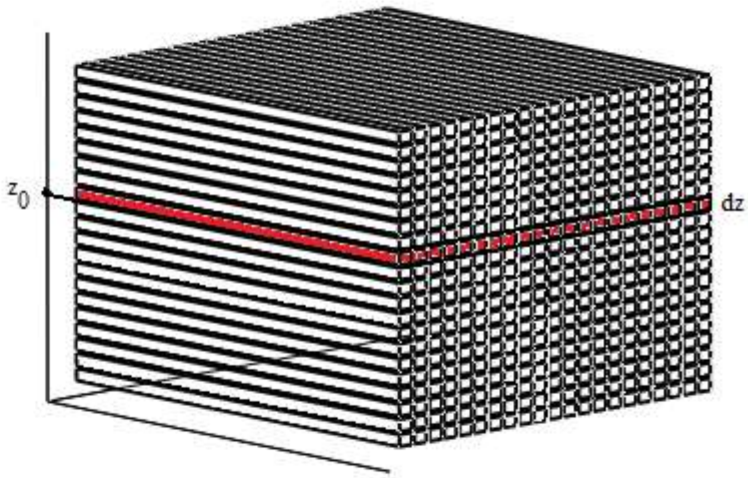
When we discussed double integrals we could graph both the function and the domain.

Let's start simply by considering a cube where $0 \le x \le 1$ $0 \le y \le 1$ $0 \le z \le 1$ .

We can partition with cross sections with constant z, with thickness dz . Then on each of these cross-sections $f = f(x,y,z_0)$ is a function of 2 variables.




Let's focus on calculating the mass of a single cross-section.




The mass of this infinitesimal cross-section is $m(z_0) = \left(\int_0^1 \int_0^1 f(x, y, z_0)\, dx dy\right)$ . In general at any z we have:

$$m(z) = \left(\int_0^1 \int_0^1 f(x, y, z)\, dx dy\right)$$

If M denotes the mass of the entire cube then m(z) = dM is an infinitesimal of the entire mass.

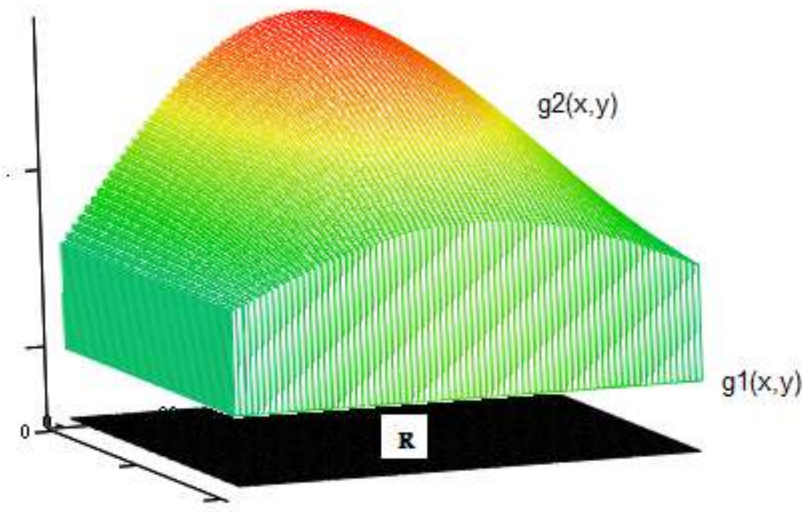
We then simply sum the masses of these cross-sections to get the entire mass M. See Animation 1.

The total Mass is the continuous sum of all these cross-sections i.e. the integral

$$M = \int_0^1 m(z)\, dz = \int_0^1 \left(\int_0^1 \int_0^1 f(x, y, z)\, dx dy\right) dz = \int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dx dy\, dz$$

This is called an iterated triple integral. Note the order of integration here is x-y-z

However typically the type of region in 3 space we consider involves a region of 3space in which we have a region between the x -y plane and a surface z = g(x,y) over a region R of the x-y plane or a region between 2 surfaces z = g1(x,y) and g2(x,y) over a region R in the x- y plane.

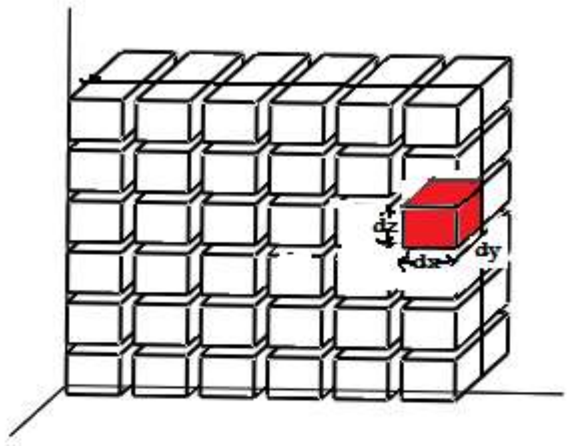



In this case the triple integral takes the following form where we usually integrate with respect to z first :

$$\iint_{R} \left[ \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right] dA$$

The outer 2 integrals are a double integral over a region R in the x- y plane.

Generally we use the symbol $\iiint f(x,y,z)dv$ where dv is the basic volume element which in rectangular coordinates is simply $dv = dzdydx$.




One way one thinking of the set up is that for the integral $\int_a^b \int_c^d \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dy\,dx$ between 2 surfaces over a rectangular domain is that in the inner integration we go from surface to surface. In the second integration we go right to left and in the outer we go from back to front. See Animation 2 .

Of course R could be any of the domains we considered in our discussion of double integrals.

In the remainder of this lecture we'll consider triple integrals in rectangular coordinates.

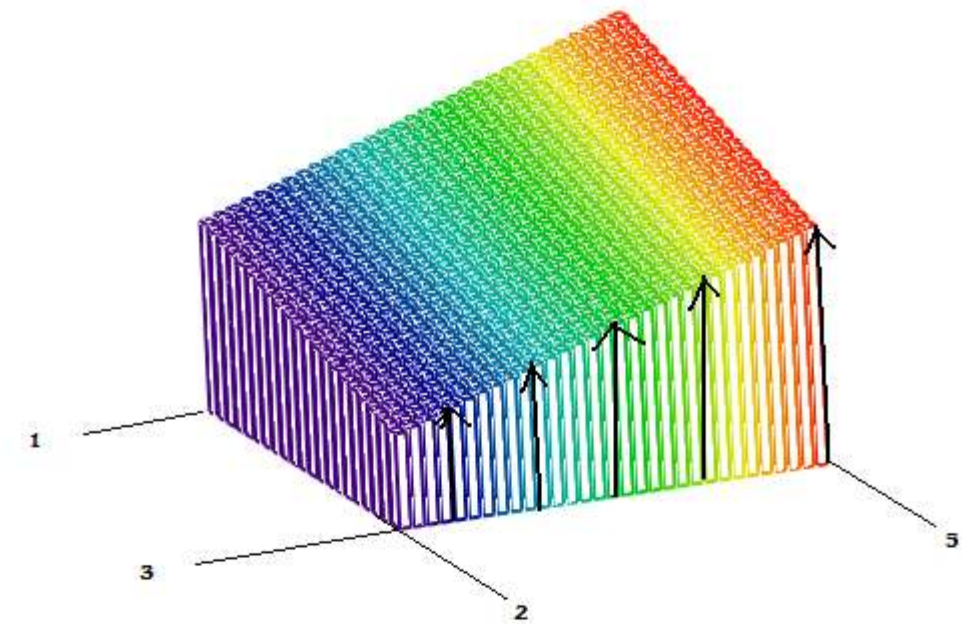
In subsequent lectures we'll consider triple integrals in Cylindrical and Spherical Coordinates.

**Example 1**

Suppose we have the surface σ given by x - y + z = 3 over the rectangle $1 \leq x \leq 3$ $2 \leq y \leq 5$ . Note z = 3-x+y.

Suppose f(x,y,z) = z is the density at any point (the mass increases linearly as z increases) . Find the mass .

See animation 3.




$$\iiint f\,dv \quad = \quad \int_1^3 \int_2^5 \int_0^{3-x+y} z\,dz\,dy\,dx$$

$$\int_1^3 \int_2^5 \int_0^{3-x+y} z\,dz\,dy\,dx = \frac{1}{2}\cdot \int_1^3 \int_2^5 z^2 \cdot \Big|_0^{3-x+y} dy\,dx = \frac{1}{2}\cdot \int_1^3 \int_2^5 \left[(3-x+y)^2\ dy\,dx\right.$$

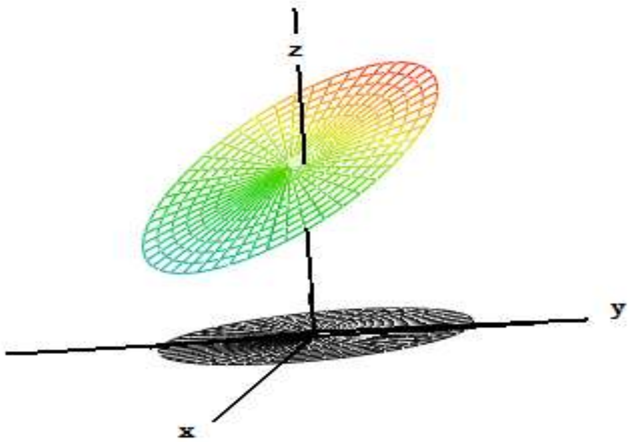
$$\frac{1}{2}\cdot \int_1^3 \int_2^5 \left[(3-x+y)^2\ dy\,dx\right. = \frac{1}{2}\cdot \int_1^3 \frac{(3-x+y)^3}{3}\cdot \Big|_2^5\ dx = \frac{1}{6}\cdot \int_1^3 (8-x)^3 - (5-x)^3\,dx$$

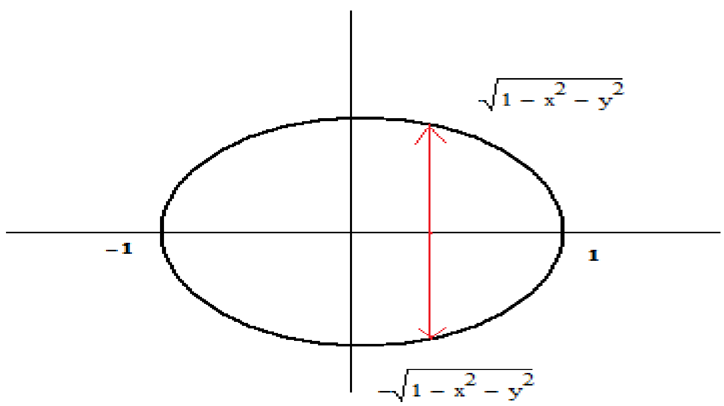
$$\frac{1}{6}\cdot \int_1^3 (8-x)^3 - (5-x)^3\,dx = \frac{1}{6}\cdot\left[-\frac{(8-x)^4}{4} + \frac{(5-x)^4}{4}\right]\cdot \Big|_1^3 = \frac{1}{24}\cdot\left[-\left(5^4-7^4\right)+\left(2^4-4^4\right)\right] = 64$$

**Example 2**

Consider Example 1 again but this time over the circular domain $x^2 + y^2 = 1$ .

Set up but do not evaluate.



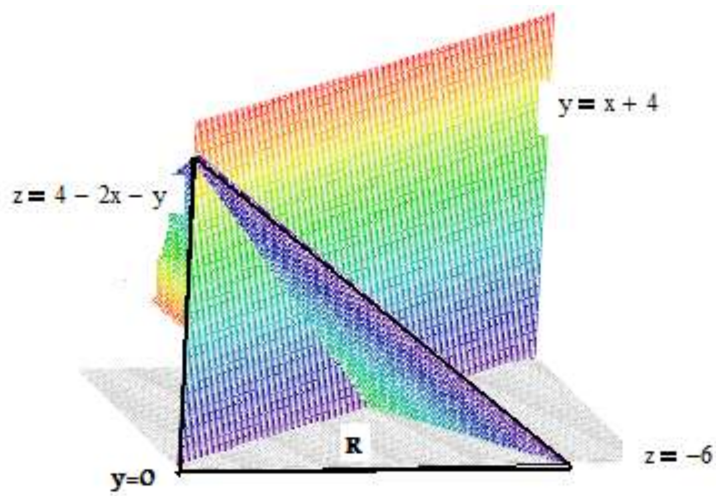




$$M = \int_{-1}^{1} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \int_{0}^{3-x+y} z\,dz\,dy\,dx$$

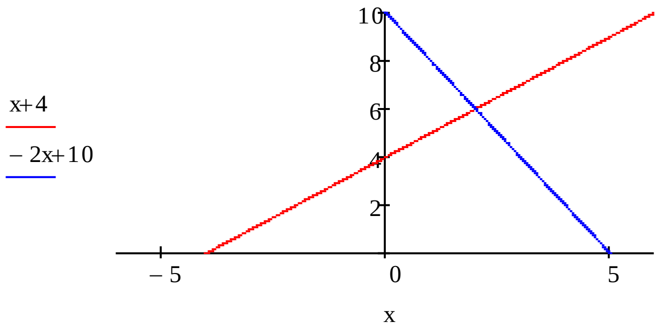
Example 3

Recall that for double integrals by taking f(x,y) = 1 we can use the double integral to calculate the area of the domain R i.e. $A = \iint dA$ , similarly we can compute the volume of a region by taking f(x,y,z) = 1 i.e. $V = \iiint dv$

Calculate the volume of the solid bounded by the vertical planes y = 0, $y = x + 4$, the horizontal plane z = - 6, and the plane $z = 4 - 2x - y$



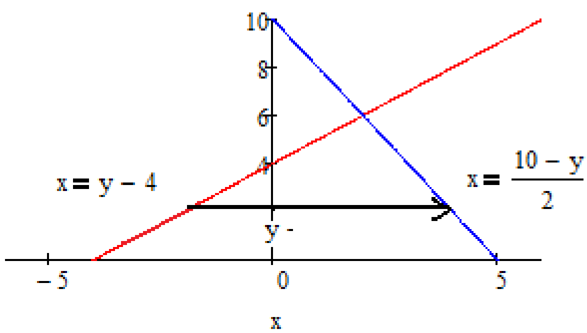

Note R is a triangular region bounded by **y = 0, y = x+4** and the intersection of z = -6 and $z = 4 - 2x - y$ or **y = - 2x +10 .**




To integrate over R we will integrate first with respect to x . ( otherwise it would take 2 integrals)

Inverting we have $x = y - 4$ and $x = \frac{10 - y}{2}$ . To find the limits of integration we set

$y - 4 = \frac{10 - y}{2}$ . This yields y = 6.




We have then $$\int_0^6 \int_{y-4}^{\frac{10-y}{2}} \int_{-6}^{4-2\cdot x-y} 1\,dz\,dx\,dy = \int_0^6 \int_{y-4}^{\frac{10-y}{2}} z\cdot \Big|_{-6}^{4-2\cdot x-y} dx\,dy$$

$$\int_0^6 \int_{y-4}^{\frac{10-y}{2}} (10 - 2x - y)\,dx\,dy = \int_0^6 \int_{y-4}^{\frac{10-y}{2}} [(10-y) - 2\cdot x]\,dx\,dy = \int_0^6 \left[(10-y)\cdot x - x^2\right]\cdot \Big|_{y-4}^{\frac{10-y}{2}} dy$$

$$\int_{0}^{6} \frac{(y-10)^2}{4} - \left(22y - 2\cdot y^2 - 56\right) dy = \int_{0}^{6} \frac{9\cdot(y-6)^2}{4}\, dy = \frac{3}{4}\cdot(y-6)^3\cdot \Bigg|_{0}^{6} = 0 - \left[\frac{-3}{4}\cdot(-6)^3\right] = 162$$