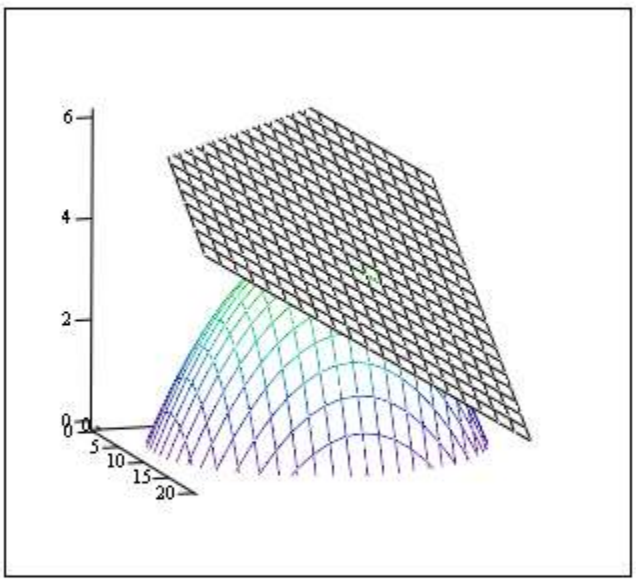
The Tangent Plane
Before Starting See the Animation on local linearity

We saw for functions of one variable the derivative is the slope of the tangent line. Further if we zoomed in on the point we saw that the function became the tangent line. This led to a discussion of the differential.

We saw in the animation that the same thing happens for functions of 2 variables, the difference being that instead of a tangent line to a curve we have the tangent plane to a surface as is seen below.
In our next lecture we'll explore the Differential for functions of 2 variables.

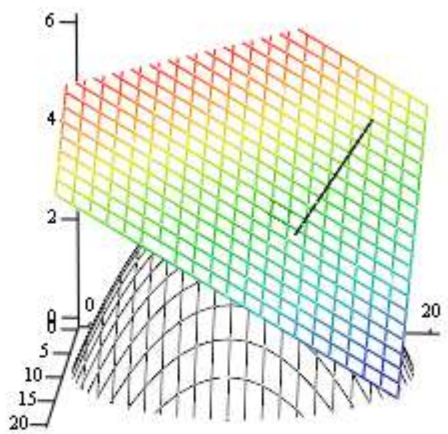
Point $(\frac{1}{2}, \frac{1}{2}, 3.5)$



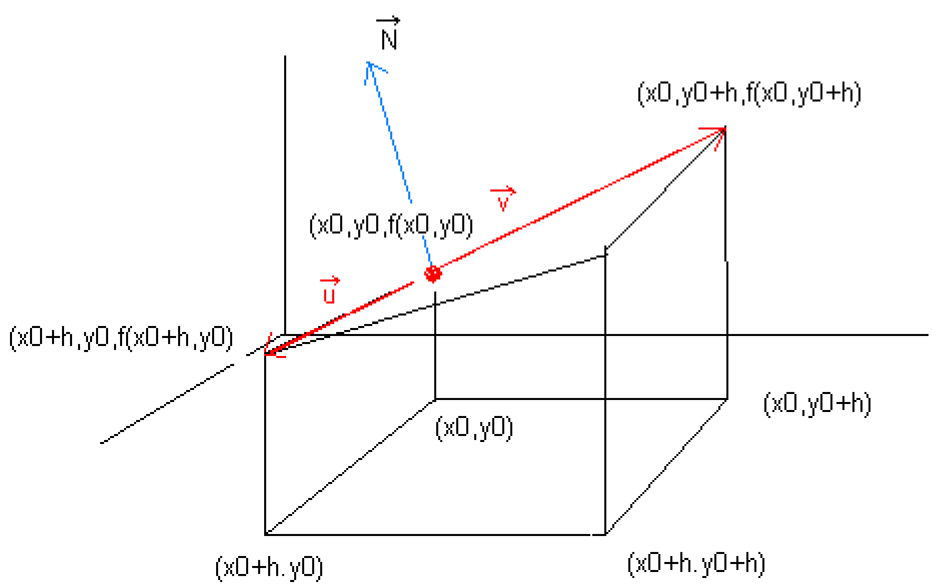
M, N

So How do we find the equation of the Tangent plane?

As with any plane we need a point and a vector perpendicular to that plane. Obviously we will know the point ($x_0, y_0, f(x_0, y_0)$) But what about the perpendicular vector?-- In the case of the tangent plane we call this the Normal to the surface.



Again as we saw by zooming in, the surface and the tangent plane are the same. So we find 2 vectors $\vec{u}$ and $\vec{v}$ lying in the tangent plane and then take their cross product to find the normal $\vec{N}$.



From the diagram we see $\vec{u} = \left[h \cdot \vec{i} + \left[\left[f\left(\left(x_0 + h, y_0 \right) - f\left(x_0, y_0 \right) \right] \cdot \vec{k} \right] \right]$ Now any vector parallel to $\vec{u}$

will do so divide by h $\vec{u} = \vec{i} + \left[\frac{f\left(\left(x_0 + h, y_0 \right) - f\left(x_0, y_0 \right) \right)}{h} \cdot \vec{k} \right.$ and letting h go to 0

$\vec{u} = \vec{i} + \frac{\delta f}{\delta x} \cdot \vec{k}$ where we are using δ for the partial differentiation operator.

Similarly $\vec{v} = \vec{j} + \frac{\delta f}{\delta y} \cdot \vec{k}$

$$\vec{N} = \vec{u} \times \vec{v} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\delta f}{\delta x} \\ 0 & 1 & \frac{\delta f}{\delta y} \end{bmatrix} = -\frac{\delta f}{\delta x}\cdot\vec{i} - \frac{\delta f}{\delta y}\cdot\vec{j} + \vec{k}$$

Recall for a plane at (x_0, y_0, z_0) with Normal $\vec{N} = a\cdot\vec{i} + b\cdot\vec{j} + c\cdot\vec{k}$ the equation is

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Therefore we obtain $z = \frac{\delta f}{\delta x}\cdot(x - x_0) + \frac{\delta f}{\delta y}\cdot(y - y_0) + z_0$ for the equation of the tangent plane .

So let's consider the example we saw in our examples above .

Find the tangent plane to $f(x,y) := 4 - x^2 - y^2$ at (1/2,1/2,3.5)

$\frac{\delta f}{\delta x} = -2\cdot x$ at (1/2,1/2,3.5) $\frac{\delta f}{\delta x} = -1$

$\frac{\delta f}{\delta y} = -2\cdot y$ at (1/2,1/2,3.5) $\frac{\delta f}{\delta y} = -1$

so $\vec{N} = \vec{i} + \vec{j} + \vec{k}$

And the equation of the tangent plane is

$$g(x,y) := 1\cdot\left(x - \frac{1}{2}\right) - 1\cdot\left(y - \frac{1}{2}\right) + 3.5$$

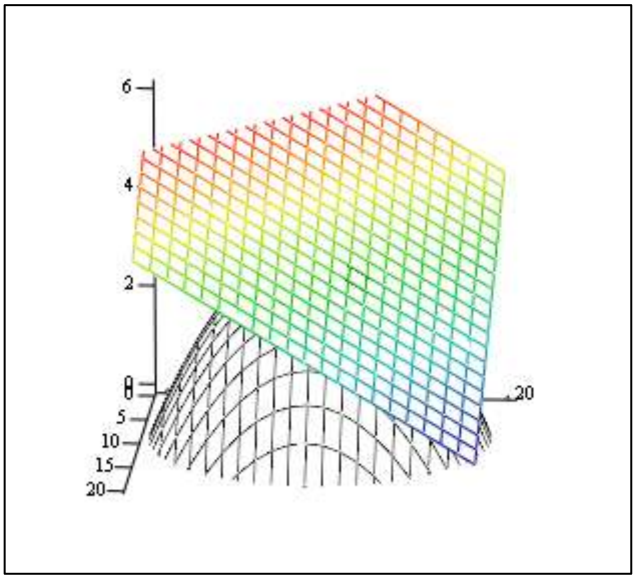
We Format the Surface:

$a := -2 \quad b := 2 \quad c := -2 \quad d := 2 \quad \Delta x := .2 \quad \Delta y := .2$

$i := 0 .. \frac{b - a}{\Delta x} \quad j := 0 .. \frac{d - c}{\Delta y} \quad x_i := a + i\cdot\Delta x \quad y_j := c + j\cdot\Delta y$

$f(x,y) := 4 - x^2 - y^2 \quad g(x,y) := 3.5 - 1\cdot\left(x - \frac{1}{2}\right) - 1\cdot\left(y - \frac{1}{2}\right)$

$M_{i,j} := f(x_i, y_j) \quad N_{i,j} := g(x_i, y_j)$



M, N

We can also compute the parametric equations of the Normal Line. The Normal Line is parallel to
$\vec{N}$

and goes through (1/2,1/2,3.5).

$x(t) = \frac{1}{2} + t \quad y(t) = \frac{1}{2} + t \quad z(t) = \frac{3}{2} + t$

For our Surface
$a := -2 \quad b := 2 \quad c := -2 \quad d := 2 \quad \Delta x := .1 \quad \Delta y := .1$

$i := 0 .. \frac{b-a}{\Delta x} \quad j := 0 .. \frac{d-c}{\Delta y} \quad x_i := a + i \cdot \Delta x \quad y_j := c + j \cdot \Delta y$

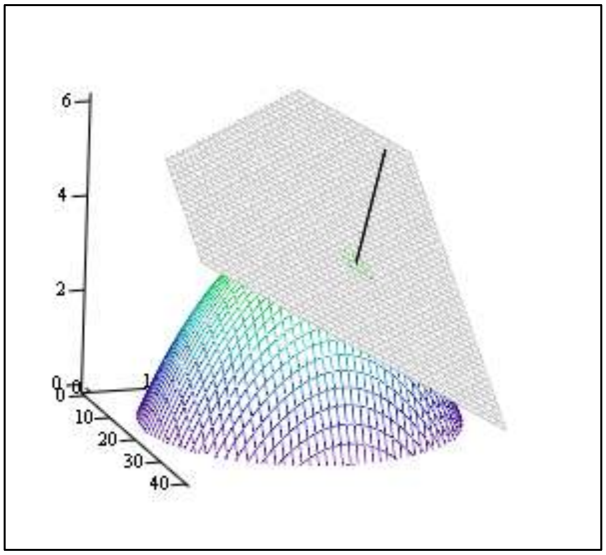
$f(x,y) := 4 - x^2 - y^2 \quad g(x,y) := 3.5 - 1 \cdot \left(x - \frac{1}{2}\right) - 1 \cdot \left(y - \frac{1}{2}\right)$

$M_{i,j} := f\left(x_i, y_j\right) \quad N_{i,j} := g\left(x_i, y_j\right)$

Now with the formatting $x_{25} = 0.5$ and $x_{25} = 0.5$ To match up our parametric equations with our surface plot we use 25 for 1/2 .

$s := 0..50 \quad t(s) := s \quad m := 0..1$

$X_{s,m} := 25 + 1 \cdot t(s) \quad Y_{s,m} := 25 + 1 \cdot t(s) \quad Z_{s,m} := 3.5 + t(s)$



M, N, (X, Y, Z)