Let $\vec{F} = (z - x)\cdot\vec{i} + (x - y)\cdot\vec{j} + e^{(x\cdot y\cdot z)}\cdot\vec{k}$ and let σ be the surface $x^2 + y^2 + z^2 = 4$ $z > 0$

Compute the circulation of $\vec{F}$ around C where C is the bounding curve of σ in the x-y plane oriented counterclockwise looking down on the surface

a. Using Stokes Theorem
b. Parameterize C and compute the line integral directly

a. Stoke's Theorem

$$\nabla \times \vec{F} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ (z - x) & x - y & e^{(x\cdot y\cdot z)} \end{bmatrix} = \left(xze^{xyz}\right)\cdot\vec{i} - \vec{j}\left(yz\cdot e^{xyz} - 1\right) + \vec{k}$$

Because the orientation is counter clockwise we use an upward Normal for the surface

$$\vec{N} = \frac{x}{z}\cdot\vec{i} + \frac{y}{z}\cdot\vec{j} + \vec{k}$$

$$\left(\nabla \times \vec{F}\right)\cdot\vec{N} = x^2\cdot e^{xyz} - y^2\cdot e^{xyz} + \frac{y}{z} + 1$$

Converting to polar coordinates:

$$\left(\nabla \times \vec{F}\right)\cdot\vec{N} = r^2\cdot\cos(2\cdot\theta)\cdot e^{\frac{r^2}{2}\cdot\sqrt{4-r^2}\cdot\sin(2\theta)} + \frac{r\cdot\sin(\theta)}{\sqrt{4 - r^2}} + 1$$

$$\int_C \vec{F}\cdot\frac{\vec{dr}}{dt}\,dt = \int\int_S \left(\nabla \times \vec{F}\right)\cdot\vec{n}\,dS = \int\int_R \left(\nabla \times \vec{F}\right)\cdot\vec{N}\,dA$$

We obtain:

$$\iint_R \left(\nabla \times \vec{F}\right)\cdot\vec{N}\,dA = \int_0^{2\cdot\pi}\int_0^2 \left(r^2\cdot\cos(2\cdot\theta)\cdot e^{\frac{r^2}{2}\cdot\sqrt{4-r^2}\cdot\sin(2\theta)} + \frac{r\cdot\sin(\theta)}{\sqrt{4-r^2}} + 1\right)\cdot r\,dr\,d\theta$$

Note the Integral

$$\int_0^{2\cdot\pi}\int_0^2 \left(r^2\cdot\cos(2\cdot\theta)\cdot e^{\frac{r^2}{2}\cdot\sqrt{4-r^2}\cdot\sin(2\theta)}\right)\cdot r\,dr\,d\theta = 0$$

To see this

1. switch the order of integration
 and use the u-substitution u = sin(2θ) and we end up with the integral or

2. simply consider the graph of cos(2θ) on the interval 0 to 2π

$$\frac{1}{2}\cdot\int_0^0 r^3\cdot e^{\frac{r^2}{2}\cdot\sqrt{4-r^2}\cdot u}\,du$$

Similarly $\displaystyle\int_0^{2\cdot\pi}\int_0^2 \left(\frac{r^2\cdot\sin(\theta)}{\sqrt{4-r^2}}\right)dr\,d\theta = 0$

We are left then with

$$\int_0^{2\cdot\pi}\int_0^{2} r\,dr\,d\theta$$

The Integral $\int_0^{2\cdot\pi}\int_0^{2} r\,dr\,d\theta$ is trivial = 4 π

2. Parameterize C

$$x(t) = 2\cdot\cos(t)$$

$$y(t) = 2\cdot\sin(t)$$

$$z(t) = 0$$

$$\overrightarrow{r(t)} = 2\cdot\cos(t)\cdot\vec{i} + 2\cdot\sin(t)\cdot\vec{j}$$

$$\vec{F} = (z - x)\cdot\vec{i} + (x - y)\cdot\vec{j} + e^{(x\cdot y\cdot z)}\cdot\vec{k} = -2\cos(t)\cdot\vec{i} + 2\cdot(\cos(t) - \sin(t))\cdot\vec{j} + \vec{k}$$

$$\frac{\overrightarrow{dr}}{dt} = -2\sin(t)\cdot\vec{i} + 2\cos(t)\cdot\vec{j}$$

$$\vec{F}\cdot\frac{\overrightarrow{dr}}{dt} = 4\cdot\sin(t)\cdot\cos(t) + 4\cdot\cos^2(t) - 4\cdot\sin(t)\cos(t) = 4\cdot\cos^2(t)$$

$$\int_C \vec{F}\cdot\frac{\overrightarrow{dr}}{dt}\,dt = \int_0^{2\cdot\pi} 4\cdot\cos^2(t)\,dt = 4\cdot\pi$$