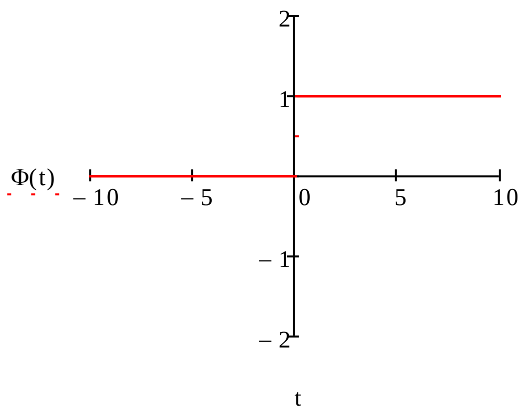
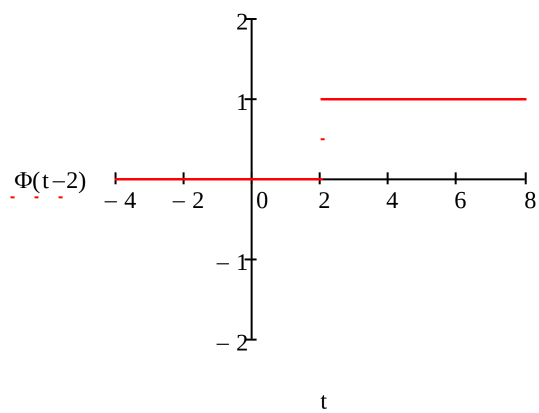
The Unit Step Function

A very simple let very useful tool in mathematics is the unit step function, denoted $\Phi(t)$ in Mathcad. The step function is also known as the Heaviside Function

$$\Phi(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



Also of great interest is its translation $\Phi(t-a)$: $\Phi(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$



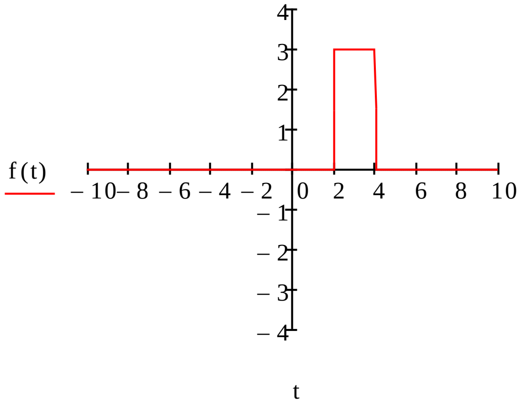
The unit step function is basically an on-off switch which is very useful in differential equations

and piecewise functions when there is a large number of pieces such as in graphs illustrating

Riemann Sums

For example suppose we want to graph a rectangular pulse of height 3 for t between 2 and 4

$f(t) := 3 \cdot \Phi(t-2) - 3 \cdot \Phi(t-4)$



Note if t < 2 $\Phi(t-2)$ and $\Phi(t-4)$ are both 0 so f(t) = 0.

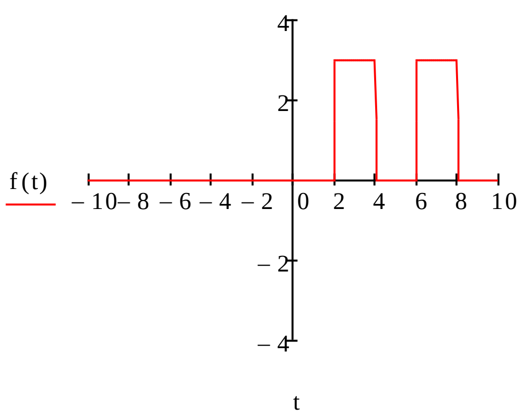
if 2 < t < 4 $\Phi(t-2)$ = 1 but $\Phi(t-4)$ = 0 so f(t) = 3

if t > 4 $\Phi(t-2)$ and $\Phi(t-4)$ are 1 and f(t) = 3 - 3 = 0

$\Phi(t-2)$ turns on the function 3 at t= 2 and $\Phi(t-4)$ turns on the function -3 at t = 4

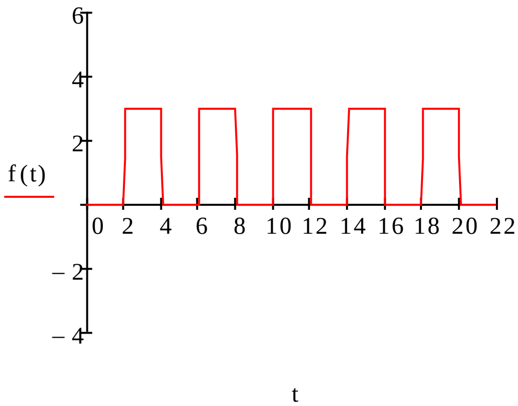
Suppose we have 2 pulses:

$f(t) := 3 \cdot \Phi(t-2) - 3 \cdot \Phi(t-4) + 3 \cdot \Phi(t-6) - 3 \cdot \Phi(t-8)$



Using $\sum$ notation we can very easily create a large number of pulses.

$$f(t) := \sum_{k=1}^{10} \left[(-1)^{k+1} \cdot 3 \cdot \Phi(t - 2 \cdot k)\right]$$



We can use the step function for any type of periodic function.

Before we examine more examples let's consider a general way of proceeding

Suppose

$$f(t) = \begin{cases} f_1(t) & \text{if } a_1 < t < a_2 \\ f_2(t) & \text{if } a_2 < t < a_3 \\ . & \\ . & \\ \left[f_{n-1}(t)\right] & \text{if } a_{n-1} < t < a_n \\ \left[f_n(t)\right] & \text{if } a_n < t \end{cases}$$

Then

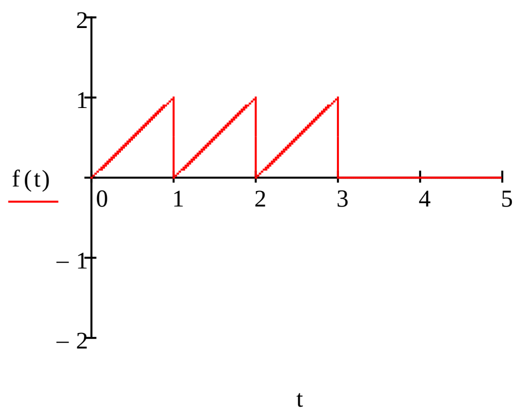
$$f(t) = \Phi\left(t - a_1\right) \cdot f_1(t) + \Phi\left(t - a_2\right) \cdot \left[f_2(t) - f_1(t)\right] + \ldots\ldots\ldots\ldots + \Phi\left(t - a_n\right) \cdot \left[f_n(t) - f_{n-1}(t)\right]$$

or the appropriate $\sum$ notation could be used.

Consider the saw tooth graph below which consists of $f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ (t-1) & \text{if } 1 < t < 2 \\ (t-2) & \text{if } 2 < t < 3 \\ 0 & \text{if } t > 3 \end{cases}$

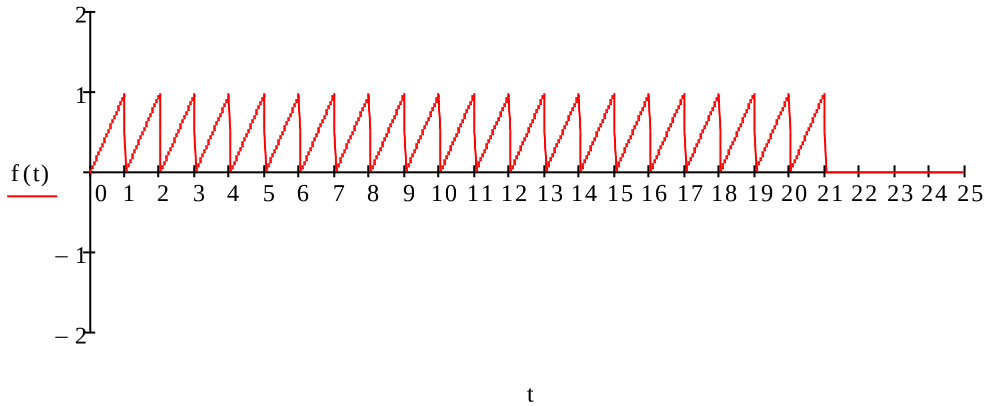
$$f(t) := \Phi(t)\cdot t + \Phi(t-1)\cdot[(t-1) - t] + \Phi(t-2)\cdot[(t-2) - (t-1)] + \Phi(t-3)\cdot[0 - (t-2)]$$

This cleans up to $f(t) := \Phi(t)\cdot t - \Phi(t-1) - \Phi(t-2) - \Phi(t-3)\cdot(t-2)$



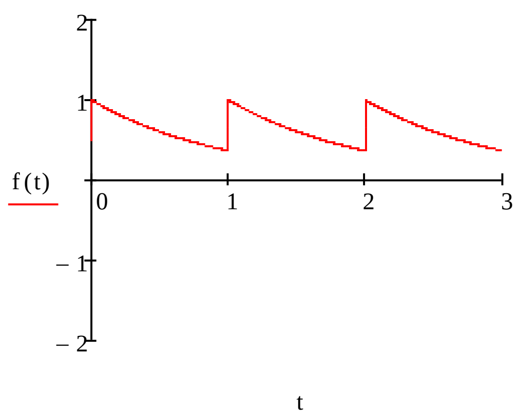
And we see we can create a large number of these saw teeth with

$$f(t) := \Phi(t)\cdot t - \left(\sum_{k=1}^{20} \Phi(t-k)\right) - \Phi(t-21)\cdot(t-20)$$



In the following example we have a decaying exponential which spikes to 1 at integer values of t then decays until t reaches the next integer:

$$f(t) := \Phi(t)\cdot e^{-t} + \Phi(t-1)\cdot\left[e^{-(t-1)} - e^{-t}\right] + \Phi(t-2)\cdot\left[e^{-(t-2)} - e^{-(t-1)}\right]$$



The following is known as the sine rectification curve essentially f(t) = |sin(t)|

Why not just use |sin(t)| ? Because for differentiation, integration or taking Laplace transforms we would have to write in piecewise form anyway

$$f(t) := \Phi(t)\cdot\sin(t) + \Phi(t-\pi)\cdot(\sin(t-\pi) - \sin(t)) + \Phi(t-2\cdot\pi)\cdot(\sin(t-2\cdot\pi) - \sin(t-\pi))$$