Triple Integrals in Spherical Coordinates

Before beginning you may want to review the spherical coordinate lab on the Computer Lab Assignments page.

In order to formulate $\iiint f dv$ in spherical coordinates we need to decide 2 things:

1. How do we set the integration limits?
2. What form does the volume element $dV$ take?

1. Setting the Integration Limits

If we want to integrate over a sphere of radius 1 $\rho$ would vary from 0 to 1, $\phi$ would vary from 0 to $\pi$ and $\theta$ would vary from 0 to $2\pi$

See Animation 1.

The integration limits would be

$$\int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{1} d\rho \, d\phi \, d\theta$$

Typically $\rho$ varies from 0 (the origin) to the outside of the surface which may either be a constant or a function $g = g(\phi, \theta)$ (usually a function of just $\phi$.)

Then $\phi$ varies between 2 angles $\alpha_1$ and $\alpha_2$

$\theta$ varies between 2 angles $\beta_1$ and $\beta_2$

Example 2 If we want the right half of a sphere we would have

See Animation 2

Example 3 If we want the upper half of a sphere we would have

See Animation 3
**Example 4** If we want the portion of a sphere in the first octant we would have 

\[ \int_0^\pi \int_0^{\pi/2} \int_0^1 r^2 \sin \theta \, dr \, d\theta \, d\phi \]

See Animation4

**Example 5** If we want the region inside the cone \( \phi = \pi / 4 \) and the sphere \( \rho = 2 \) we would have 

\[ \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 r^2 \sin \theta \, dr \, d\theta \, d\phi \]

See Animation5

2. The volume element

Again as with rectangular coordinates or cylindrical coordinates we partition according to constant values of the coordinates.

- \( \rho = \) constant --- concentric spheres
- \( \phi = \) constant -- cones
- \( \theta = \) constant vertical planes

Recall that \( r \) from cylindrical coordinates is \( r = \rho \sin(\phi) \) and \( z = \rho \cos(\phi) \)

Let's start by considering partitioning with \( \rho \) and \( \theta \)
Now we'll partition this result with respect to $\phi$ to obtain:

We have for our volume element $dV = \rho^2 \sin(\phi) \, d\rho d\phi d\theta$

In the following examples I'll concentrate on setting up the Integrals and not on their evaluations as they are relatively simple.
Example

In each of the 5 examples above Let $f(\rho, \phi, \theta) = 2 - \rho$ set up the integrals that would Calculate the mass

Example 1
\[
\int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{1} [(2 - \rho)\left(\rho^2 \cdot \sin(\phi)\right)] \, d\rho \, d\phi \, d\theta
\]
Ans: \(\frac{5\pi}{3}\)

Example 2
\[
\int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{1} [(2 - \rho)\left(\rho^2 \cdot \sin(\phi)\right)] \, d\rho \, d\phi \, d\theta
\]
Ans: \(\frac{5\pi}{6}\)

Example 3
\[
\int_{0}^{2\pi} \int_{0}^{\frac{\pi}{2}} \int_{0}^{1} [(2 - \rho)\left(\rho^2 \cdot \sin(\phi)\right)] \, d\rho \, d\phi \, d\theta
\]
Ans: \(\frac{5\pi}{6}\)

Example 4
\[
\int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \int_{0}^{1} [(2 - \rho)\left(\rho^2 \cdot \sin(\phi)\right)] \, d\rho \, d\phi \, d\theta
\]
Ans: \(\frac{5\pi}{24}\)

In example 5 Let $f(\rho, \phi, \theta) = 2 + \rho$

Example 5
\[
\int_{0}^{2\pi} \int_{0}^{\frac{\pi}{4}} \int_{0}^{2} [(2 - \rho)\left(\rho^2 \cdot \sin(\phi)\right)] \, d\rho \, d\phi \, d\theta
\]
Ans: \(\frac{28\pi}{3} \left(\sqrt{2} - 2\right)\)

Example 6

Calculate the volume of the solid in the sphere of radius 2 above the plane $z = 1$

Again we use $V = \iiint dv$

Typically when we set up a triple integral in spherical coordinates we consider a single cross-section and think of the limits on $\theta$ as rotating that cross-section about the z axis.
Considering that portion in the first octant:
Here we have first example where the variation of $\rho$ depends on $\phi$. For example if $\phi = 0$ $\rho$ varies from 1 to 2.

For an arbitrary $\phi$ the upper limit is 2. What about the lower limit? We see $\cos(\phi) = \frac{1}{\rho}$ so $\rho = \sec(\phi)$.

What about $\phi$? At the upper limit $\cos(\phi) = \frac{1}{2}$ so $\phi = \pi / 3$.

We let $\theta$ vary from 0 to $2\pi$ to rotate this section and fill out the desired volume.

We have

$$
\int_{0}^{2\pi} \int_{0}^{\pi} \int_{\sec(\phi)}^{2} \left(\rho^2 \cdot \sin(\phi)\right) d\rho \ d\phi \ d\theta
$$

We will go ahead and evaluate this one as it is slightly trickier than the ones above.

$$
\int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{2} \left(\rho^2 \cdot \sin(\phi)\right) d\rho \ d\phi \ d\theta = \frac{1}{3} \int_{0}^{2\pi} \int_{0}^{\pi} \sin(\phi) \cdot \rho^3 \ \sec(\phi) \ d\phi \ d\theta
$$

$$
\frac{8}{3} \int_{0}^{2\pi} \int_{0}^{\pi} \sin(\phi) \ d\phi \ d\theta - \frac{1}{3} \int_{0}^{2\pi} \int_{0}^{\pi} \sin(\phi) \ \sec^3(\phi) \ d\phi \ d\theta
$$

The first integral is trivial

$$
\frac{8}{3} \int_{0}^{2\pi} \int_{0}^{\pi} \sin(\phi) \ d\phi \ d\theta \rightarrow \frac{8 \pi}{3}
$$
\[
\frac{1}{3} \int_{0}^{2\pi} \int_{0}^{\pi} \sin(\phi) \sec^3(\phi) \, d\phi \, d\theta = \frac{1}{3} \int_{0}^{2\pi} \int_{0}^{\pi} \frac{\sin(\phi)}{\cos^3(\phi)} \, d\phi \, d\theta
\]

Let \( u = \cos(\phi) \) \( du = -\sin(\phi) \, d\phi \)

\[
\int_{0}^{\pi} \frac{\sin(\phi)}{\cos^3(\phi)} \, d\phi = -\int_{1}^{2} \frac{1}{u^3} \, du = \frac{1}{2} \left[ 1 \right]_{1}^{2} = 2 - \frac{1}{2} = \frac{3}{2}
\]

\[
\frac{1}{3} \int_{0}^{2\pi} \int_{0}^{\pi} \sin(\phi) \sec^3(\phi) \, d\phi \, d\theta = \frac{1}{2} \int_{0}^{2\pi} 1 \, d\theta = \pi
\]

\[
\frac{8}{3} \int_{0}^{2\pi} \int_{0}^{\pi} \sin(\phi) \, d\theta - \frac{1}{3} \int_{0}^{2\pi} \int_{0}^{\pi} \sin(\phi) \sec^3(\phi) \, d\phi \, d\theta = \frac{8\pi}{3} - \pi = \frac{5\pi}{3}
\]