Triple Integrals in Spherical Coordinates

Before beginning you may want to review the spherical coordinate lab on the Computer Lab Assignments page.

In order to formulate $\iiint f\,dv$ in spherical coordinates we need to decide 2 things:

1. How do we set the integration limits ?
2. What form does the volume element dV take ?

1. Setting the Integration Limits

If we want to integrate over a sphere of radius 1 ρ would vary from 0 to 1, ϕ would vary from 0 to π and θ would vary from 0 to 2π

See Animation 1.

The integration limits would be $\int_0^{2\cdot\pi}\int_0^{\pi}\int_0^{1} \blacksquare\, d\blacksquare\, d\blacksquare\, d\blacksquare$

Typically ρ varies from 0 (the origin) to the outside of the surface which may either be a constant or a function $g = g(\phi, \theta)$ (usually a function of just ϕ.)

Then ϕ varies between 2 angles α_1 and α_2

θ varies between 2 angles β_1 and β_2

Example 2 If we want the right half of a sphere we would have $\int_0^{\pi}\int_0^{\pi}\int_0^{1} \blacksquare\, d\blacksquare\, d\blacksquare\, d\blacksquare$

See Animation 2

Example 3 If we want the upper half of a sphere we would have $\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{2}}\int_0^{1} \blacksquare\, d\blacksquare\, d\blacksquare\, d\blacksquare$

See Animation3

Example 4 If we want the portion of a sphere in the first octant we would have $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{1} \blacksquare \, d\blacksquare \, d\blacksquare \, d\blacksquare$

See Animation4

Example 5 If we want the region inside the cone $\phi = \pi / 4$ and the sphere $\rho = 2$ we would have

$$\int_0^{2\cdot\pi} \int_0^{\frac{\pi}{4}} \int_0^{2} \blacksquare \, d\blacksquare \, d\blacksquare \, d\blacksquare$$

See Animation5

2. The volume element

Again as with rectangular coordinates or cylindrical coordinates we partition according to constant values of the coordinates.

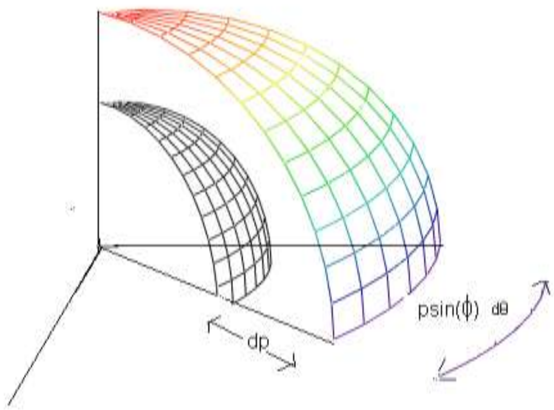
ρ =constant --- concentric spheres

ϕ = constant -- cones

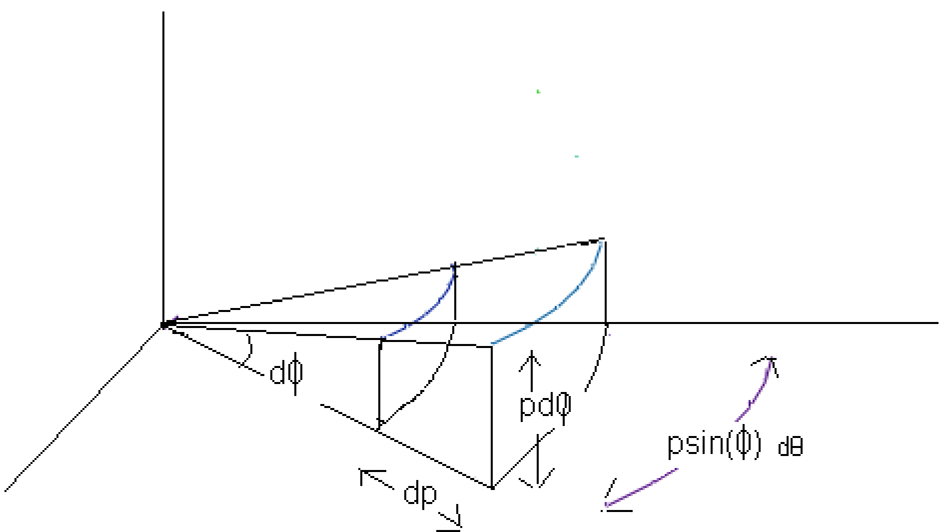
θ = constant vertical planes

Recall that r from cylindrical coordinates is $r = \rho\cdot\sin(\phi)$ and $z = \rho\cdot\cos(\phi)$

Let's start by considering partitioning with ρ and θ



Now we'll partition this result with respect to ϕ to obtain:



We have for our volume element dV = $\rho^2 \cdot \sin(\phi)\, d\rho d\phi d\theta$

In the following examples I'll concentrate on setting up the Integrals and not on their evaluations as they are relatively simple.

Example

In each of the 5 examples above Let f(ρ ,ϕ ,θ) = 2 - ρ set up the integrals that would Calculate the mass

Example1 $\int_0^{2\cdot\pi}\int_0^{\pi}\int_0^{1}\left[(2-\rho)\cdot\left(\rho^2\cdot\sin(\phi)\right)\right]d\rho\,d\phi\,d\theta$ Ans: $\frac{5\cdot\pi}{3}$

Example 2 $\int_0^{\pi}\int_0^{\pi}\int_0^{1}\left[(2-\rho)\cdot\left(\rho^2\cdot\sin(\phi)\right)\right]d\rho\,d\phi\,d\theta$ Ans: $\frac{5\cdot\pi}{6}$

Example 3 $\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{2}}\int_0^{1}\left[(2-\rho)\cdot\left(\rho^2\cdot\sin(\phi)\right)\right]d\rho\,d\phi\,d\theta$ Ans: $\frac{5\cdot\pi}{6}$

Example 4 $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{1}\left[(2-\rho)\cdot\left(\rho^2\cdot\sin(\phi)\right)\right]d\rho\,d\phi\,d\theta$ Ans: $\frac{5\cdot\pi}{24}$

In example 5 Let f(ρ ,ϕ ,θ) = 2 + ρ

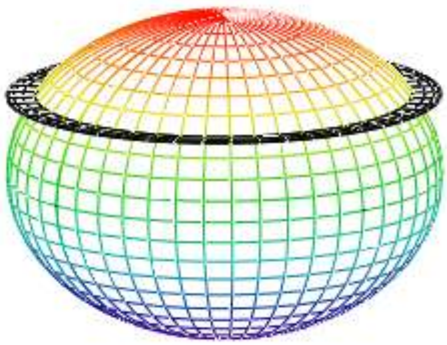
Example 5 $\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{4}}\int_0^{2}\left[(2-\rho)\cdot\left(\rho^2\cdot\sin(\phi)\right)\right]d\rho\,d\phi\,d\theta$ Ans: $\frac{28\cdot\pi\cdot\left(\sqrt{2}-2\right)}{3}$

Example 6

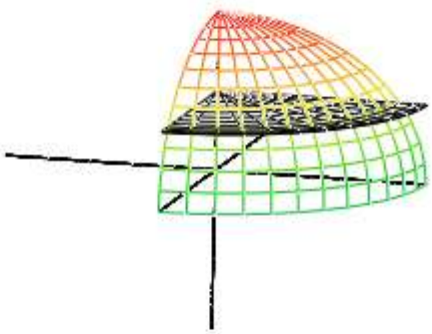
Calculate the volume of the solid in the sphere of radius 2 above the plane z = 1

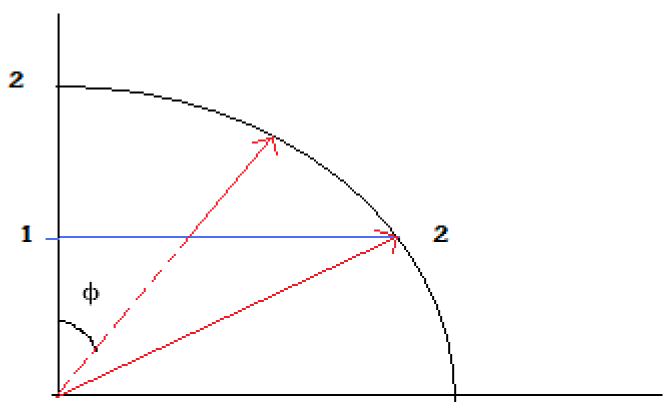
Again we use V = ∭ dv

Typically when we set up a triple integral in spherical coordinates we consider a single cross-section and think of the limits on θ as rotating that cross-section about the z axis.



Considering that portion in the first octant:





Here we have first example where the variation of ρ depends on ϕ . For example if $\phi = 0$ ρ varies from 1 to 2.

For an arbitrary ϕ the upper limit is 2. What about the lower limit ? We see $\cos(\phi) = \frac{1}{\rho}$ so $\rho = \sec(\phi)$.

What about ϕ ? At the upper limit $\cos(\phi) = \frac{1}{2}$ so $\phi = \pi/3$.

We let θ vary from 0 to 2π to rotate this section and fill out the desired volume.

We have $\displaystyle\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}}\int_{\sec(\phi)}^{2}\left(\left(\rho^2\cdot\sin(\phi)\right)\right)d\rho\,d\phi\,d\theta$

We will go ahead and evaluate this one as it is slightly trickier than the ones above .

$$\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}}\int_{\sec(\phi)}^{2}\left(\left(\rho^2\cdot\sin(\phi)\right)\right)d\rho\,d\phi\,d\theta = \frac{1}{3}\cdot\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}}\sin(\phi)\cdot\rho^3\cdot\Big|_{\sec(\phi)}^{2}\,d\phi\,d\theta$$

$$\frac{8}{3}\cdot\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}}\sin(\phi)\,d\phi\,d\theta - \frac{1}{3}\cdot\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}}\sin(\phi)\sec^3(\phi)\,d\phi\,d\theta$$

The first integral is trivial $\displaystyle\frac{8}{3}\cdot\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}}\sin(\phi)\,d\phi\,d\theta \rightarrow \frac{8\cdot\pi}{3}$

$$\frac{1}{3}\cdot\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}} \sin(\phi)\sec^3(\phi)\,d\phi\,d\theta = \frac{1}{3}\cdot\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}} \frac{\sin(\phi)}{\cos(\phi)^3}\,d\phi\,d\theta$$

Let $u = \cos(\phi)$ $du = -\sin(\phi)\cdot d\phi$

$$\int_0^{\frac{\pi}{3}} \frac{\sin(\phi)}{\cos(\phi)^3}\,d\phi = -\int_1^{\frac{1}{2}} \frac{1}{u^3}\,du = \frac{1}{2\cdot u^2}\cdot\Bigg|_1^{\frac{1}{2}} = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\frac{1}{3}\cdot\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}} \sin(\phi)\sec^3(\phi)\,d\phi\,d\theta = \frac{1}{2}\cdot\int_0^{2\cdot\pi} 1\,d\theta = \pi$$

$$\frac{8}{3}\cdot\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}} \sin(\phi)\,d\phi\,d\theta - \frac{1}{3}\cdot\int_0^{2\cdot\pi}\int_0^{\frac{\pi}{3}} \sin(\phi)\sec^3(\phi)\,d\phi\,d\theta = \frac{8\cdot\pi}{3} - \pi = 5\cdot\frac{\pi}{3}$$