

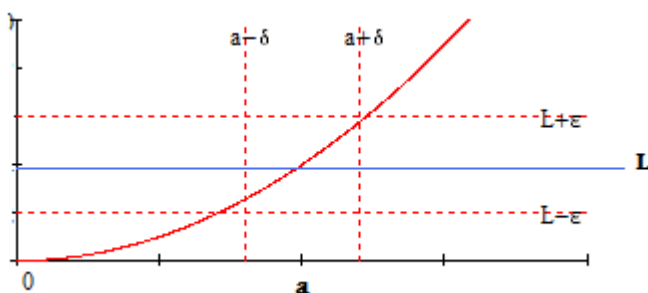
Limits - A Rigorous Approach

By $\lim_{x \rightarrow a} f(x) = L$ we mean given any $\varepsilon > 0$ there exists a $\delta > 0$ such

that

$$-\varepsilon < f(x) - L < \varepsilon \quad \text{whenever } a - \delta < x < a + \delta .$$

It is usually stated $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$.



In the animation below we show $\lim_{x \rightarrow 2} x^2 = 4$ by taking $\delta < \varepsilon / 5$ for any given ε .

So why is $\delta < \varepsilon / 5$?

$$\text{Here } |f(x) - L| = |x^2 - 4| = |(x - 2) \cdot (x + 2)| \quad \text{We can take } \delta < 1 \text{ so } x + 2 < 5.$$

This follows since if $\delta < 1$ then $x < 3$ so $x + 2 < 5$



In our example $|x - a| = |x - 2|$ So if $\delta < \varepsilon / 5$ then

$$|x^2 - 4| = |(x - 2) \cdot (x + 2)| < \frac{\varepsilon}{5} \cdot 5 = \varepsilon$$

To summarize given any ε if $|x - 2| < \varepsilon / 5$ then $|x^2 - 4| < \varepsilon$ which is precisely what we mean

by $\lim_{x \rightarrow 2} x^2 = 4$. To be more exact we should take $\delta = \min\{1, \varepsilon / 5\}$.