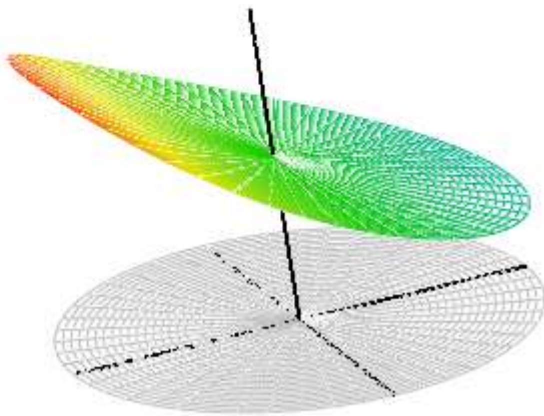
Double Integrals in Polar Coordinates

In the lecture on double integrals over non-rectangular domains we used to demonstrate the basic idea with graphics and animations the following:



However this particular example didn't show up in the examples. The function here is $f(x,y) = e^{-y}$

over the circle $x^2 + y^2 = 9$. Had we set up the integral we would have had:

$$\int\int f(x,y)\,dA = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} e^{-y}\,dy\,dx$$ uh no thanks.

However if we reformulate the problem in terms of polar coordinates we have something much more manageable .

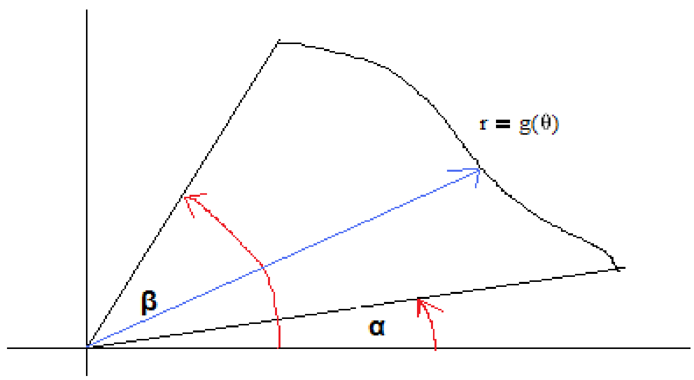
What form does $\int\int f(x,y)\,dA$ take in polar coordinates ?

There are 2 issues we must decide

1. How do we set the integration limits ?
2. What form does the areal element dA take--- the answer is ***not*** simply drdθ as you might first expect

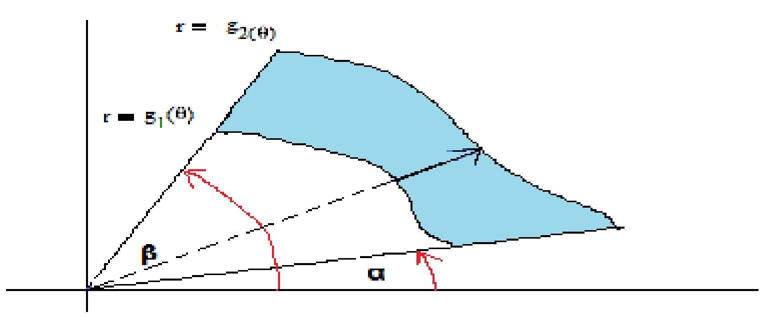
1. **Setting the Integration Limits**

Typically when polar coordinates are used we have a region such as:



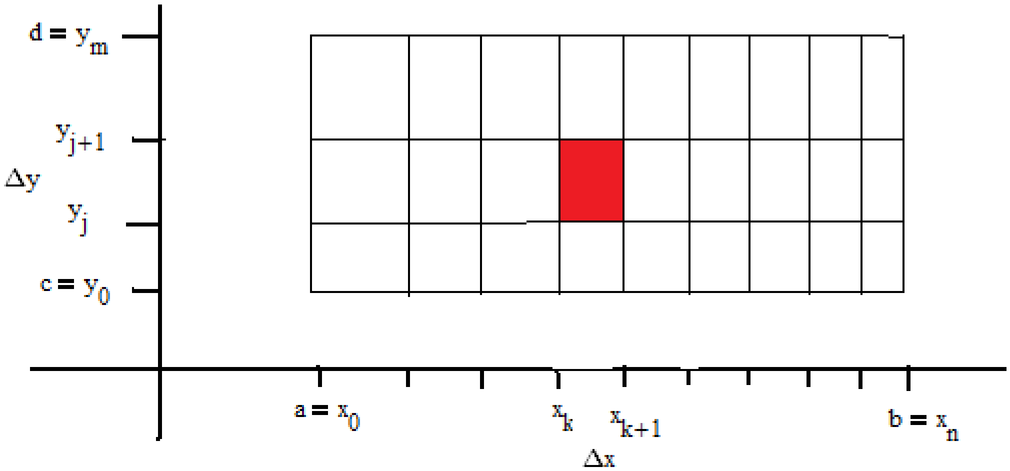
We have r varying from 0 to g(θ) and then θ varies from α to β . So the limits of integration are $\int_{\alpha}^{\beta}\int_{0}^{g(\theta)} \mathbf{f(r,\theta)}\ \mathbf{dA}$

If we have a region between 2 curves $g_1(\theta)$ and $g_2(\theta)$ we have $\int_{\alpha}^{\beta}\int_{g_1(\theta)}^{g_2(\theta)} \mathbf{f(r,\theta)}\ \mathbf{dA}$



2. The Areal element dA

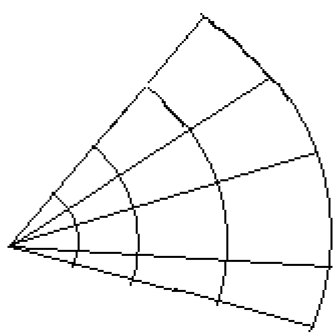
Recall in Rectangular coordinates we partition the region R according to x = constant- vertical lines and y = constant which are horizontal lines.



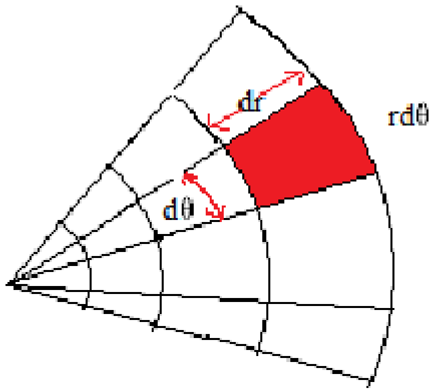
This partition creates rectangles with $\Delta A = \Delta x \Delta y$ which when we let the number of rectangles go to ∞

We obtain $dA = dxdy$

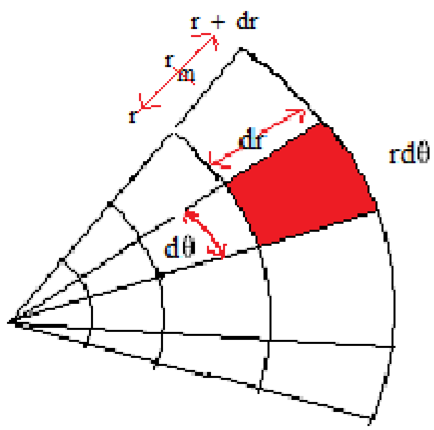
Suppose we partition the region R according to θ = constant which are lines emanating from the origin and r = constant which are concentric circles centered at the origin.



Recall the length of a circular arc is $s = r\theta$



Speaking without much rigor we have a "rectangle" whose length and width are dr and rdθ and hence dA = rdrdθ . This is a good way of remembering dA but let's be more rigorous.



dA is the difference between the area of the circular sector of radius r and the one of r +dr. Recall the area of a circular sector subtended by an angle θ is $A = \frac{1}{2}r^2 \cdot \theta$. If we let r_m denote the midpoint of the segment from r to r+dr

We have :

$$dA = \frac{1}{2} \cdot \left(r_m + \frac{dr}{2}\right)^2 \cdot d\theta - \frac{1}{2} \cdot \left(r_m - \frac{dr}{2}\right)^2 \cdot d\theta$$

$$dA = \frac{1}{2} \cdot \left(r_m\right)^2 d\theta + \frac{1}{2} r_m dr d\theta + \frac{1}{8} dr^2 d\theta - \left[\frac{1}{2} \cdot \left(r_m\right)^2 d\theta - \frac{1}{2} r_m dr d\theta + \frac{1}{8} dr^2 d\theta\right]$$

$$dA = \frac{1}{2}\cdot(r_m)^2 d\theta + \frac{1}{2} r_m dr d\theta + \frac{1}{8} dr^2 d\theta - \left[\frac{1}{2}\cdot(r_m)^2 d\theta - \frac{1}{2} r_m dr d\theta + \frac{1}{8} dr^2 d\theta\right]$$

$dA = r_m \cdot dr d\theta$ so in general we have $dA = r dr d\theta$.

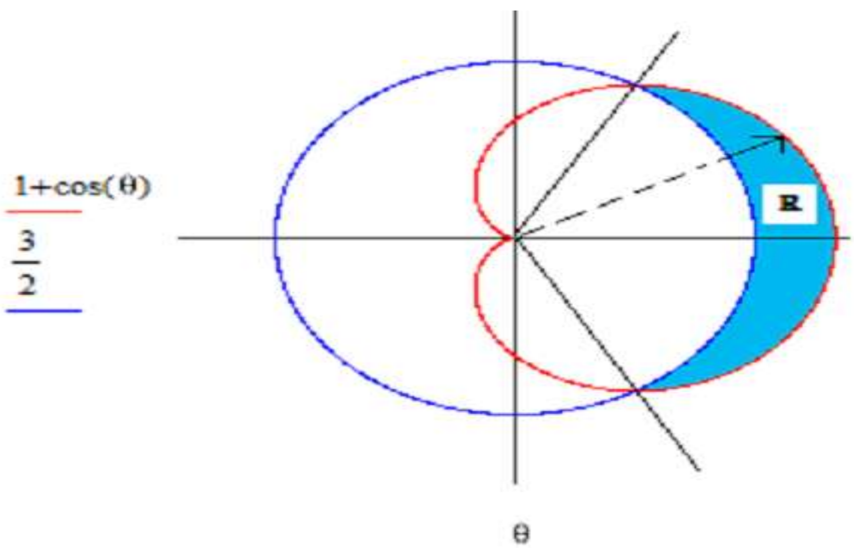
Therefore in Polar Coordinates The general form of the double Integral is :

$$\int\int f(x,y)\, dA \quad = \int_{\alpha}^{\beta}\int_{g_1(\theta)}^{g_2(\theta)} f(\theta)\cdot r\, dr\, d\theta$$

<u>Example 1</u>

Suppose we have the region inside the Cardioid $r = 1 + \cos(\theta)$ but outside the circle $r = \frac{3}{2}$.

Suppose f(x,y) = $\frac{1}{\sqrt{x^2 + y^2}}$ is the density. Find the Mass. in polar from f(r,θ) = $\frac{1}{r}$



$$\int_{\alpha}^{\beta}\int_{\frac{3}{2}}^{1+\cos(\theta)} \frac{1}{r}\cdot r\, dr\, d\theta$$

Now α and β are determined from the intersections of the 2 curves $1 + \cos(\theta) = \frac{3}{2}$

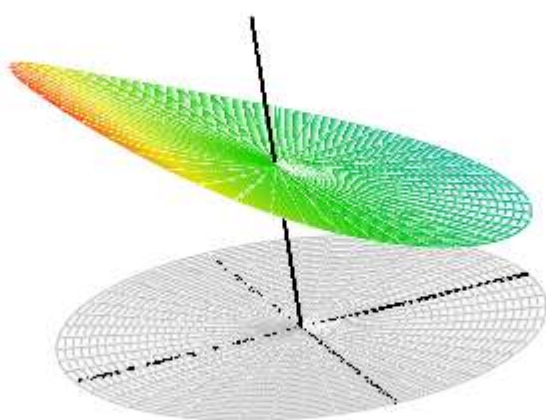
This yields $\cos(\theta) = \frac{1}{2}$ therefore θ = -π /3 and π /3

$$\int_{\alpha}^{\beta}\int_{\frac{3}{2}}^{1+\cos(\theta)} \frac{1}{r}\cdot r\,dr\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{\frac{3}{2}}^{1+\cos(\theta)} 1\,dr\,d\theta = \int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \cos(\theta) - \frac{1}{2}\,d\theta = \left(\sin(\theta) - \frac{\theta}{2}\right)\cdot \Bigg|_{\frac{-\pi}{3}}^{\frac{\pi}{3}}$$

$$\left(\sin(\theta) - \frac{\theta}{2}\right)\cdot \Bigg|_{\frac{-\pi}{3}}^{\frac{\pi}{3}} = \frac{\sqrt{3}}{2} - \left(\frac{-\sqrt{3}}{2}\right) - \frac{\pi}{6} - \left[-\left(\frac{-\pi}{6}\right)\right] = \sqrt{3} - \frac{\pi}{3} = .685$$

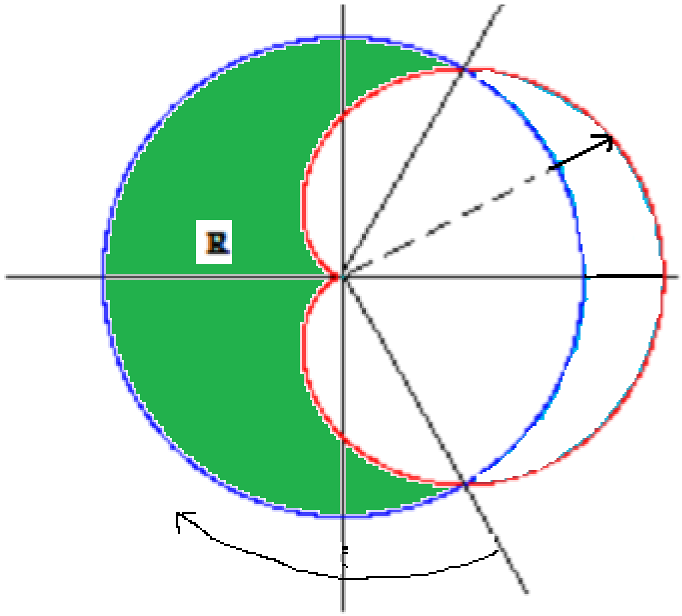
A couple of questions arise

1. Could we have simply intergrated from 0 to π /3 and double the result ? In general no. Consider the graph at the beginning of this discussion. Even though the areas in the 1st and 4th quadrants are symmetric the surface lying over these regions isn't. We need both symmetry over the region of integration and the surface over that region.



Having said that, in our example f(r,θ) =1/r is symmetric and so in our example we could have integrated half the region and doubled it.

2. Could we have set the integration limits to be 5π /3 to π /3 ? No if we did we'd be integrating clockwise covering the following region:

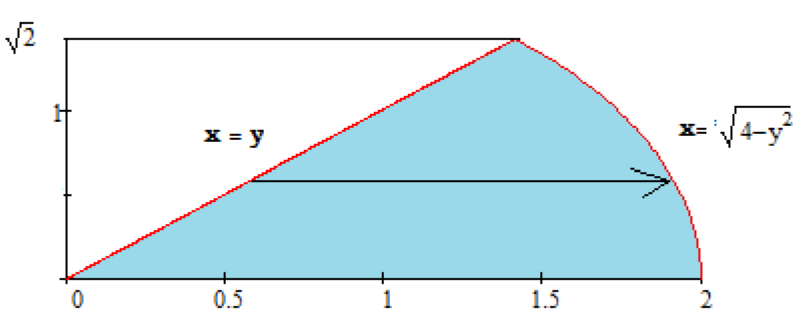


Example 2

Evaluate

$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\,dxdy$$

Let's consider a graph of the region



$x = \sqrt{4 - y^2}$ is the right half of the circle $r = 2$ and x = y is the line $\theta = \pi/4$. the 2 intersect when $\sqrt{4 - y^2} = y$ or $4 - y^2 = y^2$ or y = $\sqrt{2}$. Note since the lower limit on y is 0 we don't need to consider $-\sqrt{2}$

If we convert to polar coordinates $\dfrac{1}{\sqrt{1 + x^2 + y^2}} = \dfrac{1}{\sqrt{1 + r^2}}$

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1 + x^2 + y^2}} \, dx dy = \int_0^{\frac{\pi}{4}} \int_0^2 \frac{r}{\sqrt{1 + r^2}} \, dr \, d\theta$$

using the u-sub $u = \sqrt{1 + r^2}$ $du = \dfrac{r}{\sqrt{1 + r^2}} \cdot dr$ we obtain

$$\int_0^{\frac{\pi}{4}} \int_0^2 \frac{r}{\sqrt{1 + r^2}} \, dr \, d\theta = \int_0^{\frac{\pi}{4}} \int_1^{\sqrt{5}} 1 \, du \, d\theta = \left(\sqrt{5} - 1\right) \cdot \frac{\pi}{4} = \frac{\pi}{4} \cdot \left(\sqrt{5} - 1\right) = .971$$

Example 3

Suppose we have

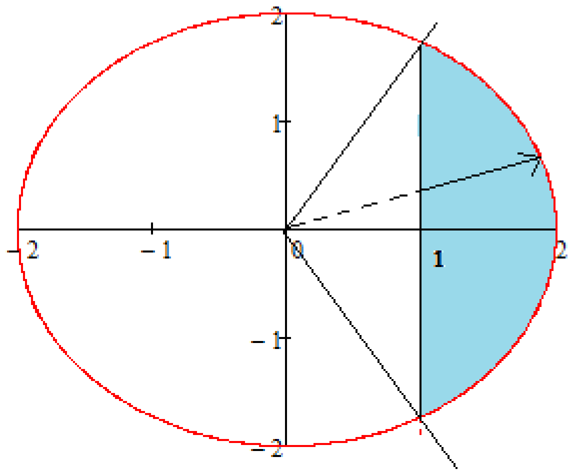
$$\int \int dA$$

Then we are looking at a solid whose height has the constant value 1 therefore the volume is numerically equal to the area of R.

i.e. we can use a double integral to compute the area of a plane region.

$$\int \int dA = A$$

With this in mind Calculate the Area of the region inside the circle r = 2 and to the right of the line x = 1.



Here we can Compute the Area in the firs quadrant and double the result.

What about x = 1

Recall x = r cos(θ)   therefore we have rcos(θ) = 1   or r = sec(θ)

Therefore r varies from sec(θ) to 2.

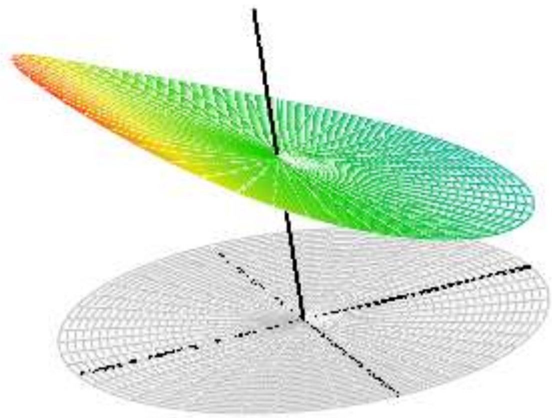
What about θ ?

At the upper limit sec(θ) = 2 from which we get θ = π /3

$$A = 2 \cdot \int_0^{\frac{\pi}{3}} \int_0^{\sec(\theta)} r \, dr \, d\theta = 2 \cdot \int_0^{\frac{\pi}{3}} \frac{\sec^2(\theta)}{2} \, d\theta = \tan(\theta) \cdot \Big|_0^{\frac{\pi}{3}} = \sqrt{3}$$

Example 4

Ok we've been putting it off as long as possible   What about $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} e^{-y} \, dy \, dx$?



We can evaluate over the first and fourth quadrant and double the result.

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} e^{-y}\,dy\,dx = 2\cdot\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{3} e^{-r\cdot\sin(\theta)}\cdot r\,dr\,d\theta$$

Which brings us to a very important point --- There is no panacea!!!!

Even though we now have another tool in or Calculus chest we will always run into problems that can't be solved analytically (you might think integrate by parts--good idea -- try it and get back to me in a year or 2).

This is the power of technology -- to get a numerical approximation just highlight the integral and hit equal

$$2\cdot\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{3} e^{-r\cdot\sin(\theta)}\cdot r\,dr\,d\theta = 74.519$$