The Line Integral and Path Independence

In the line integrals in the animations 4 - 6 we used the vector field $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j}$ and considered the line integrals around closed paths. In every case the result was 0.

The reason is that $\vec{F}$ is an example of what is known as a conservative vector field.

A conservative vector field $\vec{F} = f(x,y)\cdot\vec{i} + g(x,y)\cdot\vec{j}$ is one in which the following condition holds:

1. $\frac{\delta f}{\delta y} = \frac{\delta g}{dx}$ In our example $\frac{\delta f}{\delta y} = 1 = \frac{\delta g}{dx}$.

Why does this yield the results we obtained?

2. Theorem:

If $\vec{F}$ is conservative then $\vec{F}$ is the gradient of a scalar valued function ϕ i.e. $\vec{F} = \text{grad}\ \phi$. We call ϕ the potential function. Note in physics we use $\vec{F} = -\text{grad}\ \phi$.

This means $\vec{F} = f(x,y)\cdot\vec{i} + g(x,y)\cdot\vec{j} = \frac{\delta\phi}{\delta x}\cdot\vec{i} + \frac{\delta\phi}{\delta y}\cdot\vec{j}$

How can we exploit this? Consider the line integral along a path where at t = a the particle is at the pt (x_0,y_0) and at t = b the particle is at the pt (x_1,y_1) . $\overrightarrow{r(t)} = x(t)\cdot\vec{i} + y(t)\cdot\vec{j}$

$$\int_C f\,dx + g\,dy = \int_a^b \vec{F}\cdot\frac{\vec{dr}}{dt}\,dt$$

Recall $\vec{F} = f(x,y)\cdot\vec{i} + g(x,y)\cdot\vec{j} = \frac{\delta\phi}{\delta x}\cdot\vec{i} + \frac{\delta\phi}{\delta y}\cdot\vec{j}$ so

$$\int_a^b \vec{F}\cdot\frac{\vec{dr}}{dt}\,dt = \int_a^b \left(\frac{\delta\phi}{\delta x}\cdot\vec{i} + \frac{\delta\phi}{\delta x}\cdot\vec{j}\right)\cdot\left(\frac{dx}{dt}\cdot\vec{i} + \frac{dy}{dt}\cdot\vec{j}\right)dt = \int_a^b \left(\frac{\delta\phi}{\delta x}\cdot\frac{dx}{dt} + \frac{\delta\phi}{\delta y}\cdot\frac{dy}{dt}\right)dt = \int_a^b \frac{d\phi}{dt}\,dt = \int_{(x_0,y_0)}^{(x_1,y_1)} 1\,d\phi$$

$$\int_{(x_0,y_0)}^{(x_1,y_1)} 1\,d\phi = \phi\cdot\Big|_{(x_0,y_0)}^{(x_1,y_1)} = \phi(x_1,y_1) - \phi(x_0,y_0)$$

Which says the line integral only depends on the value of ϕ at the endpoints ? In this case we say the line integral is path independent,

This of course brings us to the question how do we find ϕ ?

But first let's summarize and consider an example.

The following statements are equivalent

1. $\vec{F}$ is conservative

2. $\frac{\delta f}{\delta y} = \frac{\delta g}{dx}$

3. $\vec{F} = \text{grad}\ \phi$

4. The line integral $\int_a^b \vec{F}\cdot\frac{\vec{dr}}{dt}\,dt$ is path independent

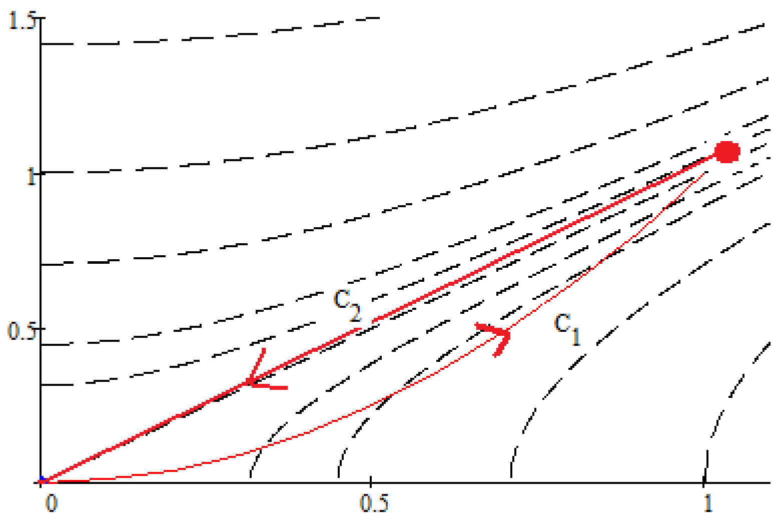
5. $$\int_a^b \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \phi(x_1,y_1) - \phi(x_0,y_0)$$

It's obvious that along a closed path the line integral is 0 since the initial and final pt are the same.

Consider the following example:

Consider the following example: $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j}$

C is made up of 2 smooth pieces one along the parabola $y = x^2$ and the other along y = x.



We found the line integral along the parabolic segment to be 1 and along y = x the result was -1.

For $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j}$ then ϕ = xy to verify this note $\frac{\delta\phi}{\delta x} = y$ and $\frac{\delta\phi}{\delta y} = x$

For the parabolic arc $$\int_a^b \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \phi(1,1) - \phi(0,0) = 1 - 0 = 1$$

For the linear segment $\int_a^b \vec{F}\cdot\frac{\vec{dr}}{dt}\,dt = \phi(0,0) - \phi(1,1) = 0 - 1 + 0$

So the only remaining question then is how do we find ϕ (x,y) ?

Since $\vec{F} = f(x,y)\cdot\vec{i} + g(x,y)\cdot\vec{j} = \frac{\delta\phi}{\delta x}\cdot\vec{i} + \frac{\delta\phi}{\delta y}\cdot\vec{j}$

It follows $f(x,y) = \frac{\delta\phi}{\delta x}$ and $g(x,y) = \frac{\delta\phi}{\delta y}$.

1 We Integrate f with respect to x adding on a function h(y) instead of an integration constant C.

$$\int \frac{\delta\phi}{\delta x}\,dx = \int f(x,y)\,dx + h(y) = \phi$$

2. Differentiate this result with respect to y and set this equal to g(x,y)

$$\frac{\delta\cdot\int f(x,y)\,dx}{\delta y} + h'(y) = g(x,y)$$

3. Solve for h(y)

For our example $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j}$ $\frac{\delta\phi}{\delta x} = y$ and $\frac{\delta\phi}{\delta y} = x$

1. $\phi(x,y) = \int \frac{\delta\phi}{\delta x}\,dx = \int y\,dx + h(y) = xy + h(y)$

2. $\frac{\delta\phi}{\delta y} = x = x + h'(y)$

3. $h'(y) = 0$ so h(y) = constant

Therefore ϕ = xy

(for purposes of evaluation we can ignore the constant just as in the definite integral for real valued functions)

Example 2

Evaluate $\int (3x - y + 1)\,dx - (x + 4\cdot y + 2)dy$ on any path from (-2,1) to (1,1)

Here $\vec{F} = (3x - y + 1)\cdot\vec{i} - (x + 4\cdot y + 2)\cdot\vec{j}$

1. Verify $\vec{F}$ is conservative

$$\frac{\delta f}{\delta y} = -1 = \frac{\delta g}{\delta x}$$

2. Find ϕ

$$\frac{\delta\phi}{\delta x} = (3x - y + 1)$$

$$\phi(x,y) = \int (3x - y + 1)\,dx = 3\cdot\frac{x^2}{2} - x\cdot y + x + h(y)$$

$$\frac{\delta\phi}{\delta y} = g(x,y) = -(x + 4\cdot y + 2) = -x + h'(y)$$

$$-4\cdot y - 2 = h'(y)$$

Therefore $h(y) = -2\cdot y^2 - 2\cdot y$

It follows $\phi(x,y) = 3\cdot\frac{x^2}{2} - x\cdot y + x - 2\cdot y^2 - 2y$

$$\int (3x - y + 1)\,dx - (x + 4\cdot y + 2)dy = \int_{(-2,1)}^{(1,1)} (3x - y + 1)\,dx - (x + 4\cdot y + 2)dy = \phi(1,1) - \phi(-2,1)$$

$$\phi(1,1) - \phi(-2,1) = -\frac{5}{2} - 2 = \frac{-9}{2}$$

Before considering our last example we want to consider line integrals in 3-space.

Recall we have

The following statements are equivalent

1. $\vec{F}$ is conservative

2. $\frac{\delta f}{\delta y} = \frac{\delta g}{dx}$

3. $\vec{F} = \text{grad } \phi$

4. The line integral $\int_a^b \vec{F} \cdot \frac{\overrightarrow{dr}}{dt}\, dt$ is path independent

5. $\int_a^b \vec{F} \cdot \frac{\overrightarrow{dr}}{dt}\, dt = \phi(x_1, y_1) - \phi(x_0, y_0)$

It's obvious that along a closed path the line integral is 0 since the initial and final pt are the same.

What if we have a vector field in 3 spaces ?

The only real change is #2 above. If you need review the concept of the curl of a vector field on the page

"Vectors in General"

The condition $\frac{\delta f}{\delta y} = \frac{\delta g}{dx}$ is replaced by curl $\overrightarrow{\text{curl} F} = \nabla \times \vec{F} = 0$.

The curl is only defined in 3 spaces. However suppose we take a vector field $\vec{F} = f(x, y) \cdot \vec{i} + g(x, y) \cdot \vec{j}$ in 2 spaces

and add a 0 $\vec{k}$ component to create a vector field in 3space.

$$\nabla \times \vec{F} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta \cdot}{dx} & \frac{\delta \cdot}{dy} & \frac{\delta \cdot}{dz} \\ f & g & 0 \end{pmatrix} = \left(\frac{\delta f}{\delta y} - \frac{\delta g}{dx}\right) \cdot \vec{k}$$

If $\frac{\delta f}{\delta y} = \frac{\delta g}{dx}$ then $\nabla \times \vec{F} = 0$ and $\vec{F}$ is conservative . When $\nabla \times \vec{F} = 0$ We say the vector field is irrotational.

To summarize we have the following are equivalent statements:

1. $\vec{F}$ is conservative

2. $\frac{\delta f}{\delta y} = \frac{\delta g}{dx}$ in 2 space or $\nabla \times \vec{F} = 0$ in 3space

3. $\vec{F}$ is irrotational

4. $\vec{F} = \text{grad } \phi$

5. The line integral $\int_a^b \vec{F} \cdot \frac{\overrightarrow{dr}}{dt} \, dt$ is path independent

6. $\int_a^b \vec{F} \cdot \frac{\overrightarrow{dr}}{dt} \, dt = \phi(x_1, y_1) - \phi(x_0, y_0)$ or $\int_a^b \vec{F} \cdot \frac{\overrightarrow{dr}}{dt} \, dt = \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0)$ in 3 space.

Example 3

Evaluate $\displaystyle\int \frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz$ along any path from (1,1,1) to (2,3,4)

$$\vec{F} = \frac{1}{x}\cdot\vec{i} + \frac{1}{y}\cdot\vec{j} + \frac{1}{z}\cdot\vec{k}$$

$$\nabla \times \vec{F} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta\cdot\blacksquare}{dx} & \frac{\delta\cdot\blacksquare}{dy} & \frac{\delta\cdot\blacksquare}{dz} \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{pmatrix} = 0$$

$$\frac{\delta\phi}{\delta x} = \frac{1}{x} \qquad \frac{\delta\phi}{\delta y} = \frac{1}{y} \qquad \frac{\delta\phi}{\delta z} = \frac{1}{z}$$

$$\phi = \int \frac{1}{x}\,dx + h(x,y) = \ln(x) + h(y,z)$$

$$\frac{\delta\phi}{\delta y} = \frac{1}{y} = \frac{\delta h(y,z)}{\delta y}$$

$$h(y,z) = \int \frac{1}{y}\,dy + k(z) = \ln(y) + k(z)$$

$$\phi = \ln(x) + \ln(y) + k(z)$$

$$\frac{\delta\phi}{\delta z} = \frac{1}{z} = k'(z)$$

$$k(z) = \ln(z)$$

$$\phi = \ln(x) + \ln(y) + \ln(z)$$

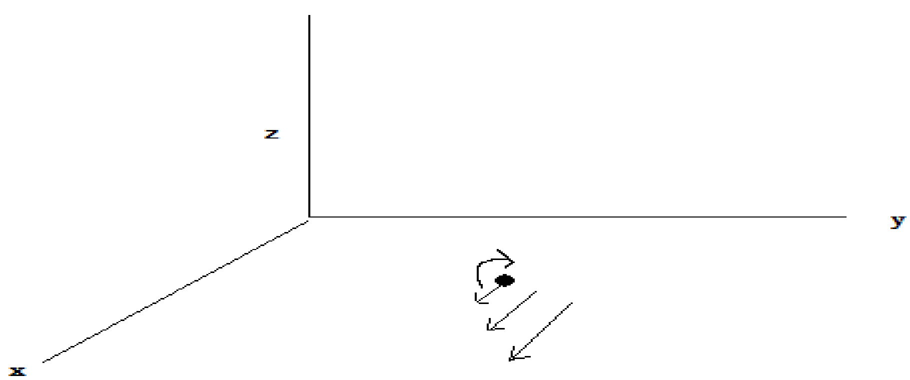
$$\int \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = \phi(2,3,4) - \phi(1,1,1) = \ln(2) + \ln(3) + \ln(4) = 3.178$$

One Final Point -- If $\nabla \times \vec{F} = 0$ Why do we say $\vec{F}$ is irrotational ?

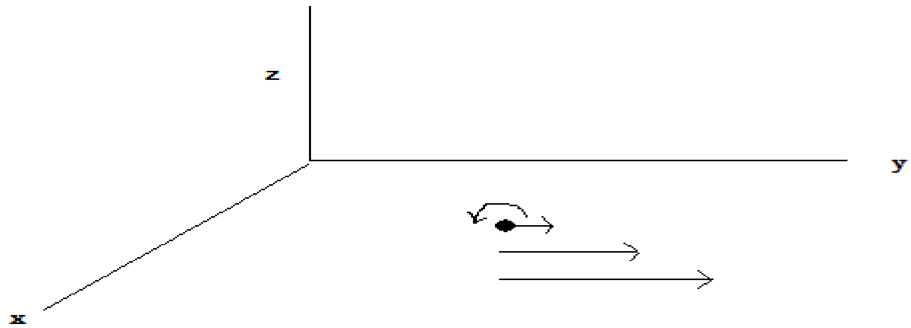
Consider the k component $\left(\frac{\delta f}{\delta y} - \frac{\delta g}{dx}\right)\cdot\vec{k}$ (You can do a similar analysis for the i and j components)

We'll focus on a single point .

f is the x component of the vector field. suppose at a point $\frac{\delta f}{\delta y} > 0$. This would produce a clockwise rotation about a pt.



g is the y component of the vector field. suppose at the same point $\frac{\delta g}{\delta x} > 0$. This would produce a counter clockwise rotation about a pt.



If $\frac{\delta g}{\delta x} = \frac{\delta f}{\delta y}$ then the rotations would cancel and hence no rotation.

It might be instructive for you to consider the i and j components of $\nabla \times \vec{F}$ to see the same type of cancellation.