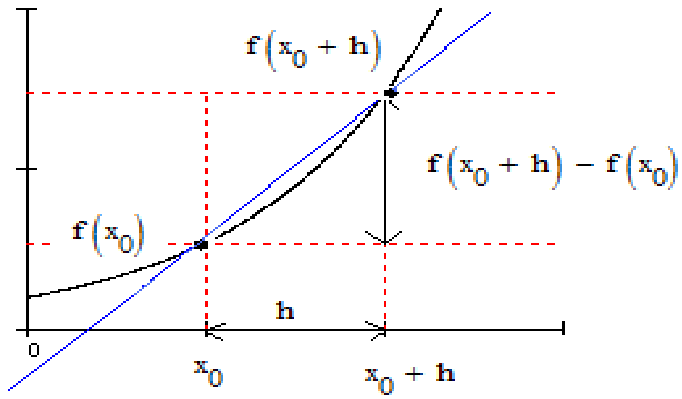
Partial Derivatives

Recall that for functions of one variable to define the derivative at a point $(x_0, f(x_0))$ we start with the secant line through the points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$.



The slope of this secant line $\frac{f(x_0 + h) - f(x_0)}{h}$ is the average rate of change.

We then define the derivative as the limit of this average rate of change as h goes to 0:

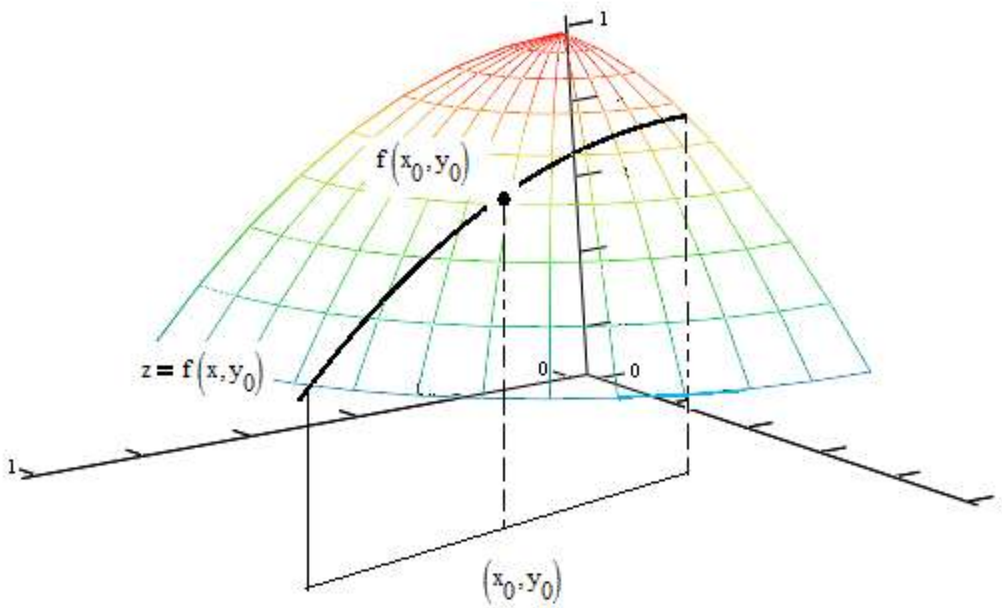
$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} .$$ See the Animation Tangent Line

For functions of 2 variables we use this basic idea however starting at a point (x, y) the rate at which $f(x, y)$ is changing depends on which direction we travel from (x, y) in the domain. This is further complicated by the fact there are an infinity of directions we can travel from (x, y). Instead of talking about the derivative we talk about directional derivatives. (See the page on this website devoted to this topic.)

However, all directional derivatives can be formulated in terms of just 2 derivatives--the partial derivatives--which is the topic of this discussion.

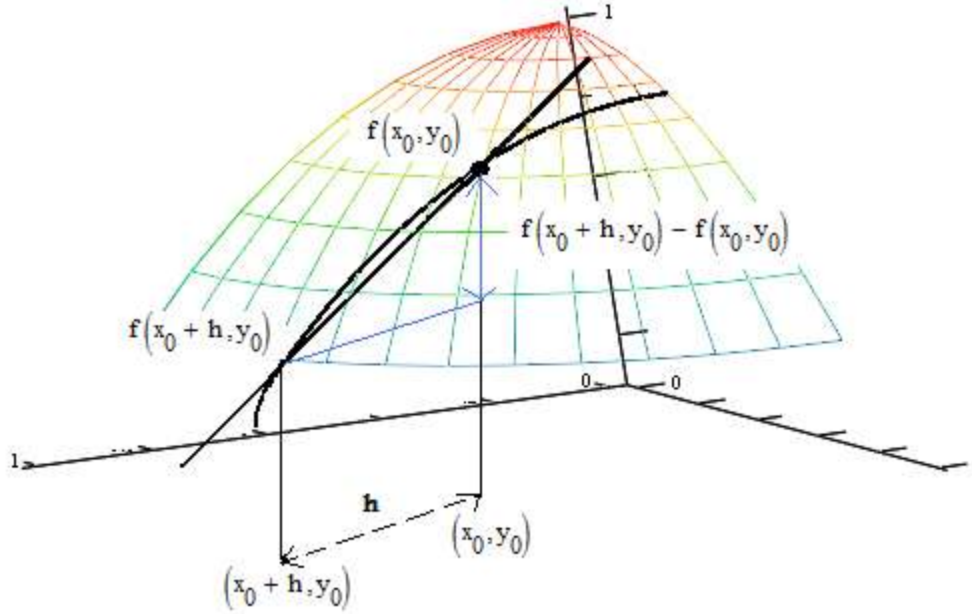
Partial Derivatives

Let z = f(x.y) be a surface and let (x_0, y_0, z_0) be a point on that surface . If we intersect the surface with the plane y = y_0 then we obtain a curve in the surface $z = f(x, y_0)$ which is essentially a function of x only.



We now proceed as we did for functions of one variable. We add a secant line through the points $(x_0, y_0, f(x_0, y_0))$

and $(x_0 + h, y_0, f(x_0 + h, y_0))$.



Now we let h go to 0 and note we have the tangent line to the curve $z = f(x, y_0)$ at the point $(x_0, y_0, f(x_0, y_0))$.

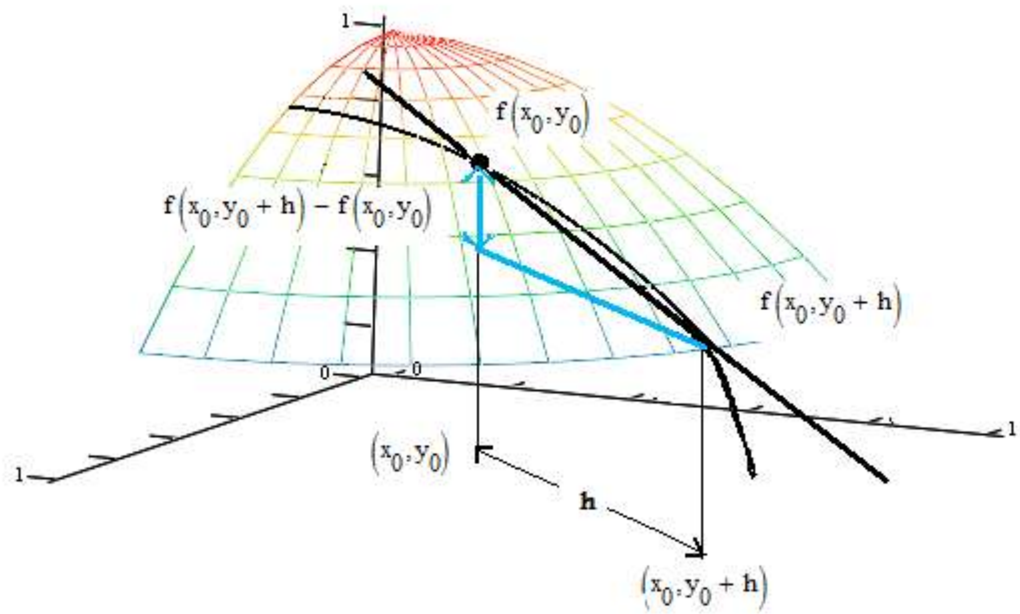
See the Animation - Partial Derivative with respect to x.

With this in mind we define the partial derivative with respect to x,symbolized by ∂ f/∂ x or f_x :

$$\partial f/\partial x = f_x = \lim_{h \to 0} \frac{\left(f(x_0 + h, y_0) - f(x_0, y_0)\right)}{h} .$$

Similarly we can intersect $z = f(x,y)$ with the plane $x = x_0$ and again add a secant line through the points

$\left(x_0, y_0, f\left(x_0, y_0\right)\right)$ and $\left(x_0, y_0 + h, f\left(x_0, y_0 + h\right)\right)$.



Again we let h go to 0 and note we have the tangent line to the curve $z = f\left(x_0, y\right)$ at the point $\left(x_0, y_0, f\left(x_0, y_0\right)\right)$

[See the Animation - Partial Derivative with respect to y.](#)

As before with this in mind we define the partial derivative with respect to y, symbolized by ∂ f/∂ y or f_y :

$$\partial f/\partial y = f_y = \lim_{h \to 0} \frac{\left(f\left(x_0, y_0 + h\right) - f\left(x_0, y_0\right)\right)}{h}.$$

In Calculus1 after the first week or so you developed techniques of differentiation and hardly ever used the limit definition. We'll do the same here.

An important point not to miss is that to Calculate $\partial f/\partial y$ we simply treat x as a constant using the same techniques as we learned in Calculus 1. Similarly to Calculate $\partial f/\partial x$ we simply treat y as a constant.

Examples

1. Let $f(x,y) = x \cdot y^2 \cdot \sin(x \cdot y)$ Calculate $\partial f/\partial x$ and $\partial f/\partial y$

$\partial f/\partial x =$ $y^2 \cdot \sin(x \cdot y) + x \cdot y^2 \cdot (y \cdot \cos(xy)) = y^2 \cdot (\sin(xy + xy \cdot \cos(xy))$

$\partial f/\partial y =$ $2xy \cdot \sin(x \cdot y) + x \cdot y^2 \cdot (x \cdot \cos(xy)) = x \cdot \left(2y\sin(xy) + xy^2 \cdot \cos(xy)\right)$

2. Let $f(x,y) = x \cdot e^{-y} + \cos(x) \cdot \ln(y)$ Calculate f_x and f_y

$f_x = e^{-y} - \sin(x) \cdot \ln(y)$

$f_y = -xe^{-y} + \frac{\cos(x)}{y}$

Now that we have the Partial Derivative we can start exploring some of the implications

See the pages on this website:

1. Directional Derivatives and Gradient

2. The Tangent Plane and Differential for f(x,y) (analog to the tangent line and differential for functions of a single variable)

3. Flux Integrals and Surface Integrals

4. Line Integrals (specifically as pertains to Green's Theorem

5. Optimization for functions of 2 variables