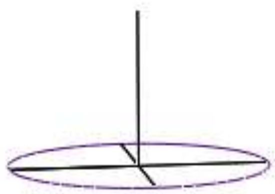
Parametric Surface Plots

We saw in the labs on graphs in cylindrical and spherical coordinates examples of what are known as parametric surface plots. In this lab we are going to consider the idea of parametric surface plots in general.

We saw that when we discussed parametric curves in 2 - space we can generate a circle with equation $x^2 + y^2 = 1$ using the

equations $x(t) := \cos(t)$ and $y(t) := \sin(t)$



Further we saw that in 3 space the equation $x^2 + y^2 = r^2$ actually represents a cylinder as z can take on any value. For example if we want a cylinder where z varies from 0 to 1 we would have:

The equations that would represent this cylinder would be :

$x(t) := \cos(t)$

$y(t) := \sin(t)$ $\quad 0 \le t \le 2\pi$

$z(s) := s$ $\quad 0 \le s \le 1$

Note this differs from a curve defined parametrically in that we need 2 parameters s and t to generate a surface whereas with parametric equations we need only one parameter to generate a curve .

This is called a parametric surface plot -- Using 2 parameters to create a surface.

We can think of this in 2 ways

1. We generate the circle $x^2 + y^2 = 1$ and then let z vary to create the cylinder

See Animation 1.

2. We can allow z to vary from 0 to 1 and then rotate this 360 deg about the z axis

See animation 2

You may be thinking why not use

$x(t) := \cos(t)$

$y(t) := \sin(t)$

$z(t) := t$

But this would generate the helix a curve in 3 space

So the question becomes then how do we format the computer to generate parametric surface plots.

Formatting the Computer

Suppose we want to graph the cylinder $x^2 + y^2 = 1$ from above

1. Define the range on u and v $u := 0..63$ $v := 0..10$ Note s and t must be integers

2. We define two more variables u and v so we can use any increment we choose-- here 0.1

$t(u) := \frac{u}{10}$ $s(v) := \frac{v}{10}$ (We saw the same thing when we defined parametric curves)

3. We define our parametric equations

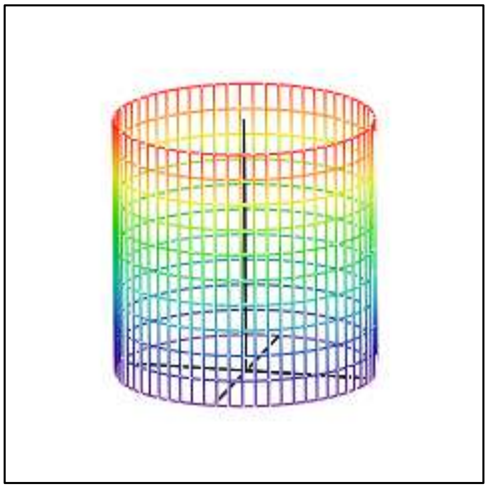
$x_{u,v} := \cos(t(u))$ $y_{u,v} := \sin(t(u))$ $z_{u,v} := s(v)$

We'll also add the coordinate axes:

$i := 0..1$ $l := 0..40$

$X1_{i,l} := 0$ $Y1_{i,l} := 0$ $Z1_{i,l} := (l - 10) \cdot 1$

Just as in the case of parametric equations we then use a 3- D surface plot with the vector (x,y,z) in the place holder :



$(x, y, z), (X1, Y1, Z1), (Y1, Z1, X1), (Z1, X1, Y1)$

Example 2

Suppose we want the surface $x^2 + y^2 = 1$ but cut by the plane z = y +1

We begin as before:

1. $u := 0..63$ $v := 0..10$

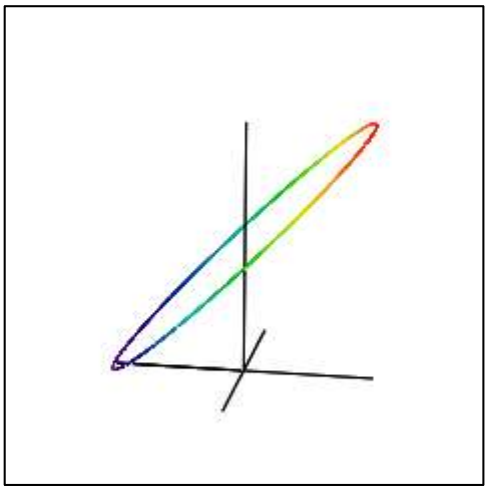
2. $t(u) := \frac{u}{10}$ $s(v) := \frac{v}{10}$

3. $x_{u,v} := \cos(t(u))$ $y_{u,v} := \sin(t(u))$

But how do we define z?

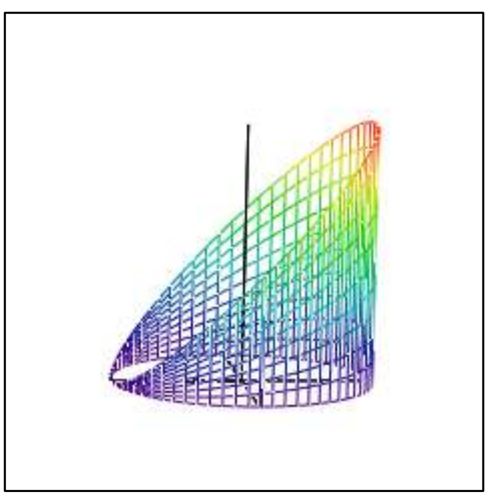
We might be tempted to use $z_{u,v} := y_{u,v} + 1$

But again we would only generate the bounding curve of the upper surface:



$(x, y, z), (X1, Y1, Z1), (Y1, Z1, X1), (Z1, X1, Y1)$

We want z to vary from 0 to y+1 so we use $z_{u,v} := s(v) \cdot \left(y_{u,v} + 1\right)$ See Animation 3



$(x, y, z), (X1, Y1, Z1), (Y1, Z1, X1), (Z1, Y1, X1)$

So the proper formatting is :

1. $u := 0..63$ $v := 0..10$

2. $t(u) := \frac{u}{10}$ $s(v) := \frac{v}{10}$

3. $x_{u,v} := \cos(t(u))$ $y_{u,v} := \sin(t(u))$

$z_{u,v} := s(v) \cdot \left(y_{u,v} + 1\right)$

Exercises

Exercise 1

Graph the cylinder $x^2 + z^2 = 9$ between the planes y = -1 and y =1.

Exercise 2

Graph the horn

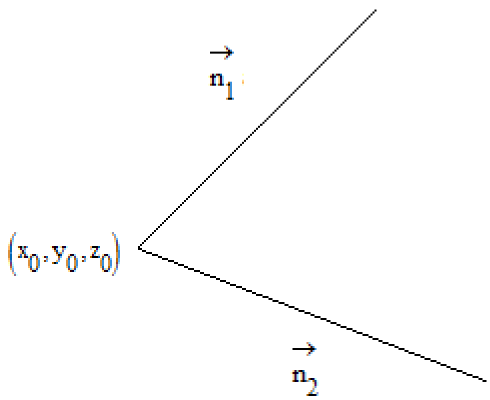
$x = t$ $y = \frac{1}{t^2 + 1}\cos(s)$ $z = \frac{1}{t^2 + 1}\sin(s)$ $0 \le t \le 3$ $0 \le s \le 2\pi$

<u>Planes Defined as Parametric Surface Plots</u>

Suppose we want to graph the triangle (1,0,0), (1,2,0), and (3,0,0)

In general how do we graph a plane as a parametric surface plot ?

Suppose we have a point $\left(x_0, y_0, z_0\right)$ and 2 non parallel vectors $\vec{n}_1$ and $\vec{n}_2$ which is precisely what we need to generate a plane .



$$\vec{n}_1 = a_1 \cdot \vec{i} + b_1 \cdot \vec{j} + c_1 \cdot \vec{k}$$

$$\vec{n}_2 = a_2 \cdot \vec{i} + b_2 \cdot \vec{j} + c_2 \cdot \vec{k}$$

Then we have:

$$x = x_0 + s \cdot a_1 + t \cdot a_2$$

$$y = y_0 + s \cdot b_1 + t \cdot b_2$$

$$z = z_0 + s \cdot c_1 + t \cdot c_2$$

What we are saying here is for each fixed s we create a vector parallel to $\vec{n}_2$. As s then varies we create the entire plane

See Animation 4

Let's return then to our original question

Suppose we want to graph the triangle with vertices (1,0,0), (1,2,0), and (0,0,3)

There are 2 ways to proceed here

1. Graph the plane and restrict the axes - simplest but least mathematical
2. Develop set of parametric equations which graph only the triangle--a little more mathematically elegant

1. A simple approach

To get our 2 vectors we consider the vector from (1,0,0) to (1,2,0) which is $2 \cdot \vec{j}$

and the vector from (1,2,0) to (0,0,3) which is $-\vec{i} - 2 \cdot \vec{j} + 3 \cdot \vec{k}$

For our fixed point we choose (1,0,0) both s and t vary from 0 to 1

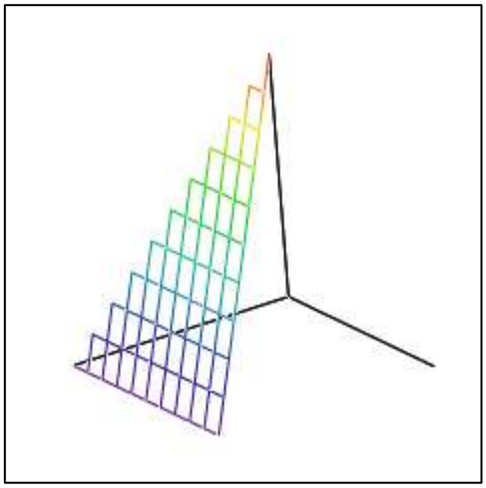
$x = 1 - t$ $\quad\quad y = 2 \cdot s - 2 \cdot t$ $\quad\quad z = 3t$

$u := 0..10$ $\quad\quad v := 0..10$

$s(u) := \frac{u}{10}$ $\quad\quad t(v) := \frac{v}{10}$

$x2_{u,v} := 1 - t(v)$ $\quad\quad y2_{u,v} := 2 \cdot s(u) - 2t(v)$ $\quad\quad z2_{u,v} := 3 \cdot t(v)$

Note with the given parameterization we are actually generating a parallelogram. However if we restrict our axes such that $0 \leq x \leq 1$, $0 \leq y \leq 2$ and $0 \leq z \leq 3$ we get the graph of the triangle we want.



$(x2, y2, z2), (X1, Y1, Z1), (Y1, Z1, X1), (Z1, X1, Y1)$

2. We start by obtaining the eqn of the plane $3x + z = 3$

For Fixed x in the xzplane z = -3x + 3 so we have the vector from (x,0,0) to (x,0,3-3x)

For fixed x in the xy plane y = 2x so we have the vector from (x,0,0) to (x, 2x,0)

We generate the graph by fixing t and allowing s to vary from 0 to 1 . We then allow t to vary from 0 to 1.This creates a segment parallel to the xz plane and then extends in the y direction.
See Animation 5.

$x = s$

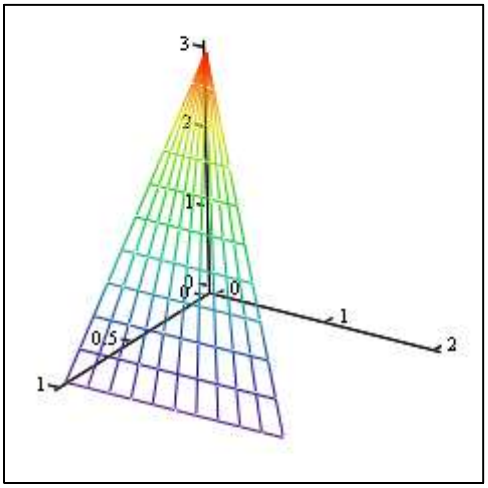
$y = 2s \cdot t$

$z = -3s + 3$

$u := 0..10$ $v := 0..10$

$s(u) := \frac{u}{10}$ $t(v) := \frac{v}{10}$

$x3_{u,v} := s(u)$ $y3_{u,v} := 2 \cdot s(u) \cdot t(v)$ $z3_{u,v} := -3 \cdot s(u) + 3$



(x3, y3, z3)

Exercise

Graph the triangle with vertices (1,0,0), (0,1,0), and (0,0,1) in the 2 ways described above