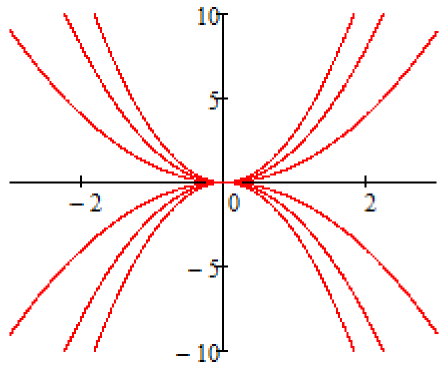
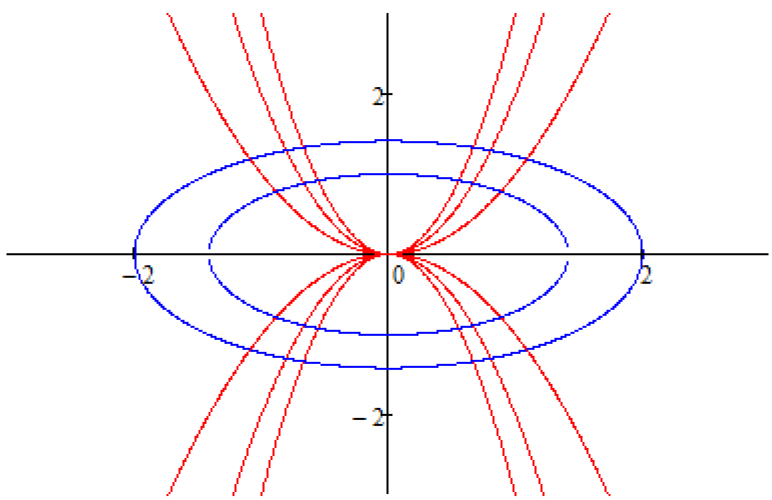
Orthogonal Trajectories

Suppose we are given a family of curves such as the one below:



For example these could represent the streamlines of a flow field or the field lines of a Force field. By the Orthogonal trajectories to this family of curves we mean the family of curves such that at each point of intersection the tangent lines to the vector field are perpindicular to the tangentlines of the orthognal trajectories.



See Animation 1

Three questions naturally arise:

1. What is the importance of the Orthognal Trajectories

2. How do we find them?

3. How do we graph an entire family of curves in an efficient manner?

The answer to the first is that the orthogonal trajectories represent the curves in which the magnitude of the velocity or the force is the same at each point on that curve. In the case of the flow field the orthognal trajectories are called the velocity potential and in the case of Force Fileds the orthogonal trajectories are called equipotential curves--curves in which the magnitude of the Force is the same.

There are 2 ways in which we can generate the differential equation to obtain the orthognal trajectories.

Method 1

Differentiate f(x,y,λ) with respect to x and solve for dy/dx.

Using the fact that perpindicular lines have slopes which are negative reciprocals solve $\frac{dY}{dX} = \frac{-1}{\frac{dy}{dx}}$ for Y.

Example Suppose we have $x^2 + y^2 = \lambda^2$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

For the orthogonal trajectories

$$\frac{dY}{dX} = \frac{Y}{X}$$

We usually change back to x and y at this point

$$\frac{dy}{dx} = \frac{y}{x}$$

We could simply separate the variables and integrate but it is actually simpler if we recognize this as a differential equatic with homogeneous coefficients and Let z = y/x

$$z + x\cdot\frac{dz}{dx} = z$$

$$\frac{dz}{dx} = 0$$

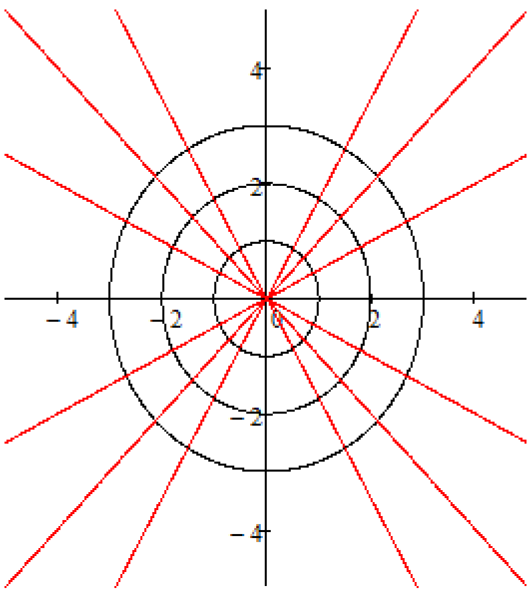
$$z = c$$

$$y = cx$$

Which is a family of lines. See Animation 2

$$f(x,\lambda) := \sqrt{\lambda^2 - x^2}$$

$$g(x,c) := c\cdot x$$



Method 2 using the differential

Recall if f = f(x,y) then the differential of f is given by $df = \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy$

If we move everything to the left hand side we have f(x,y,λ) = 0

It follows $\frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy = 0$

solve for $\frac{dy}{dx} = \frac{-\frac{\delta f}{\delta x}}{\frac{\delta f}{\delta y}}$

It follows the orthognal trajectories are the solutions to $\frac{dy}{dx} = \frac{\frac{\delta f}{\delta y}}{\frac{\delta f}{\delta x}}$

For $x^2 + y^2 = \lambda^2$ $\quad x^2 + y^2 - \lambda^2 = 0$

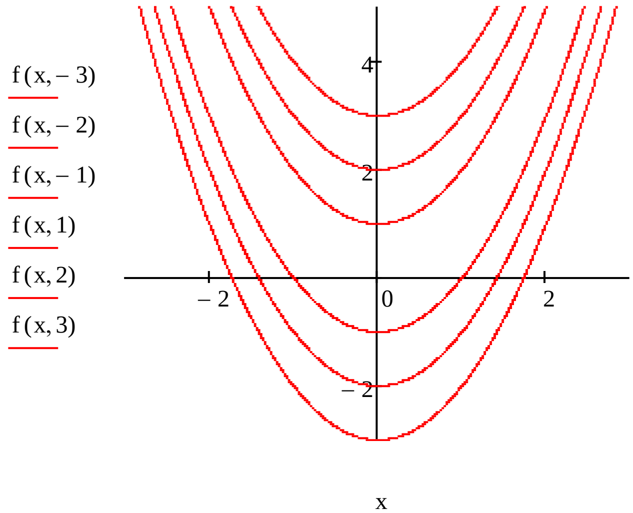
For the orthogonal trajectories $\frac{dy}{dx} = \frac{2y}{2x} = \frac{y}{x}$ which is precisely what we had before. Generally this second method involves less algebra-- but you have to understand the differential of a function of 2 variables. In some schools students take differential equations after Calculus 2 and may not be familiar with the differential. In which case method 1 will still work.

Graphing a family of curves in Mathcad.

Suppose we have the family $y = x^2 + \lambda$

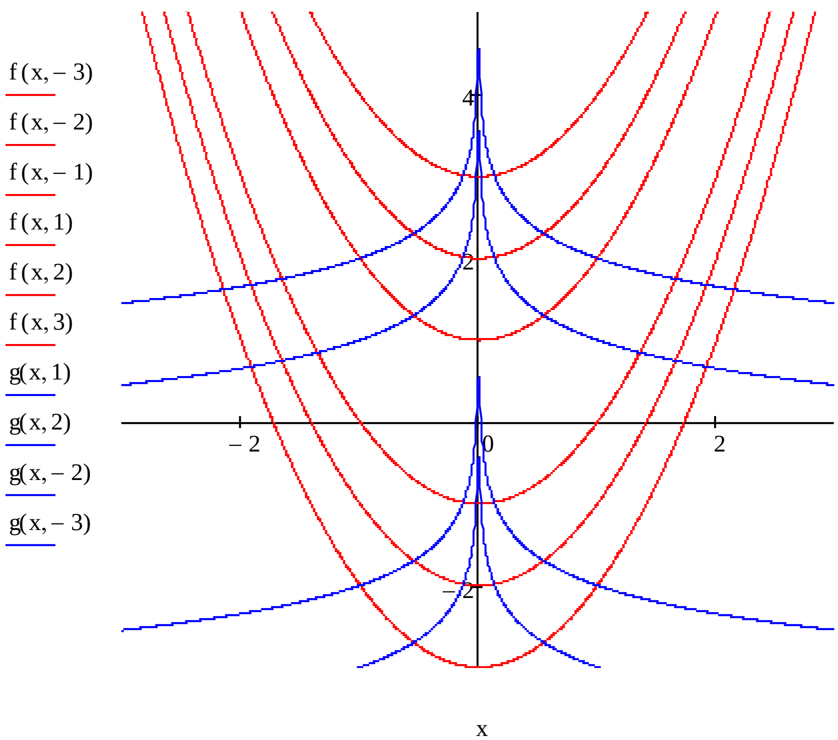
Define $f(x,\lambda) := x^2 + \lambda$ Then simply plot $f(x,\lambda)$ for various values of λ as in the graph below.

Note you only have to define one function!



Show the orthognal trajectories are the curves y = $\ln\left(\frac{1}{\sqrt{|x|}}\right) + c = \frac{-1}{2}\cdot\ln(|x|) + c$

$f(x,\lambda) := x^2 + \lambda \qquad g(x,c) := \frac{-1}{2}\cdot\ln(|x|) + c$



See Animation 3

In the introductory example we started with $y = \lambda \cdot x^2$

$$F(x, y, \lambda) = y - \lambda \cdot x^2 = 0$$

For the orthognal trajectories

$$\frac{dy}{dx} = \frac{1}{-2 \cdot \lambda \cdot x}$$

Here we must eliminate λ

$$\frac{dy}{dx} = \frac{1}{-2 \cdot \left(\frac{y}{x^2}\right) \cdot x} = \frac{x}{-2 \cdot y}$$

$$y \cdot dy = \frac{-x}{2} \cdot dx$$

$$\frac{y^2}{2} + x^2 = c$$

$$\frac{y^2}{2c} + \frac{x^2}{c} = 1$$

Which is the family of ellipses we saw in the graph and the animation.