Newton's Law of Cooling

Newton's Law of Cooling states the the rate at which the temperature of a body changes is proportional to the temperature difference between the environment and the body.

$\frac{dy}{dt} = k\cdot(T - y)$ where k is the diffusitivity and depends on the thermal properties of the body.

We are going to consider 2 types of problems

1. The temperature of the environment is constant

2. The temperature of the environment is changing

1. Constant temperature of the environment. Suppose the temperature of the environment is a constant 20C. when a body is found it has a temperature of 32C and 2 hours later it has a temperature of 27C.

a. Find y(t) the temperature of the body at time t

b. If it is known that the body has a normal temperature of 37.5 C how long had the body been dead when originally found?

We have $\frac{dy}{dt} = k\cdot(20 - y)$ $y(0) = 32$

We can solve this by separating the variables

$$\frac{dy}{(20 - y)} = k\cdot dt$$

$$\int \frac{1}{(20 - y)}\, dy = \int k\, dt$$

Note that $y = 20$ is an equiilibrium solution therefore since $y(0) = 32$ $y > 20$ for all t.

therefore $|20-y| = y-20$

$-\ln(|20 - y|) = k \cdot t + c$

$-\ln(y - 20) = k \cdot t + c$

$y - 20 = e^{c} \cdot e^{-kt} = Ce^{-kt}$

$y(t) = 20 + C \cdot e^{-k \cdot t}$

applying the initial condition $y(0) = 32 = 20 + C$

Therefore $y(t) = 20 + 12 \cdot e^{-k \cdot t}$

To find k we use $y(2) = 27$

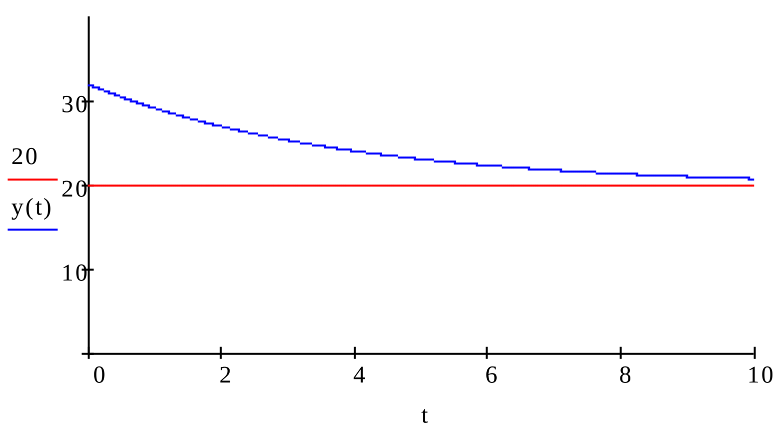
$y(2) = 27 = 20 + 12 \cdot e^{-2 \cdot k}$

$27 = 20 + 12 \cdot e^{-2 \cdot k}$

$\frac{\ln(12)}{2} - \frac{\ln(7)}{2} = k$

$k = .269$

$y(t) := 20 + 12 \cdot e^{-.269 \cdot t}$



And just as it should y(t) approaches the temperature of the environment

b. For $y(t) = 37.5$

$$20 + 12 \cdot e^{-.269 \cdot t} = 37.5$$

has solution(s)

$$-1.4025807849125206$$

Therefore the body had been dead 1.4 hrs.

Example 2 Suppose the temperature of the environment varies $\pm$ 10F sinusoidally from a mean of 55F over a 24 hour period.

Then $T(t) := 55 + 10\sin\left(\frac{\pi}{12} \cdot t\right)$.

Suppose an object is taken out of a refigerator at 45 F at t = 0. Suppose k = .15

Find y(t).

$$\frac{dy}{dt} = .15\left(55 + 10\sin\left(\frac{\pi}{12} \cdot t\right) - y(t)\right) \qquad y(0) = 45.$$

This is not separable however with a little manipulation we see it is linear and can be solved with an integrating factor. (After we solve it we'll present an alternative solution using Laplace transforms)

$$\frac{dy}{dt} + .15y = 8.25 + 1.5\sin\left(\frac{\pi}{12} \cdot t\right)$$

the integrating factor is $e^{.15 \cdot t}$

$$\frac{d\left(y \cdot e^{.15 \cdot t}\right)}{dt} = 8.25e^{.15 \cdot t} + 1.5\sin\left(\frac{\pi}{12} \cdot t\right) \cdot e^{.15 \cdot t}$$

$$y \cdot e^{.15 \cdot t} = \int 8.25 e^{.15 \cdot t} \, dt + \int 1.5 \sin\left(\frac{\pi}{12} \cdot t\right) \cdot e^{.15 \cdot t} \, dt$$

We'll let Mathcad do the integrals

$$\int 8.25 e^{.15 \cdot t} \, dt \rightarrow 55.0 e^{0.15 \cdot t}$$

$$\int 1.5 \sin\left(\frac{\pi}{12} \cdot t\right) \cdot e^{.15 \cdot t} \, dt$$

floating point evaluation yields

$$-16.5 e^{0.15 \cdot t} \cdot (0.262 \cos(0.262t) + -0.15 \sin(0.262t))$$

$$y(t) := -4.323 \cos(0.262t) + 2.475 \sin(0.262t) + 55.0 + C \cdot e^{-.15 \cdot t}$$

$$y(0) = 45 = 55 - 4.323 + C$$

$$45 = 55 - 4.323 + C$$

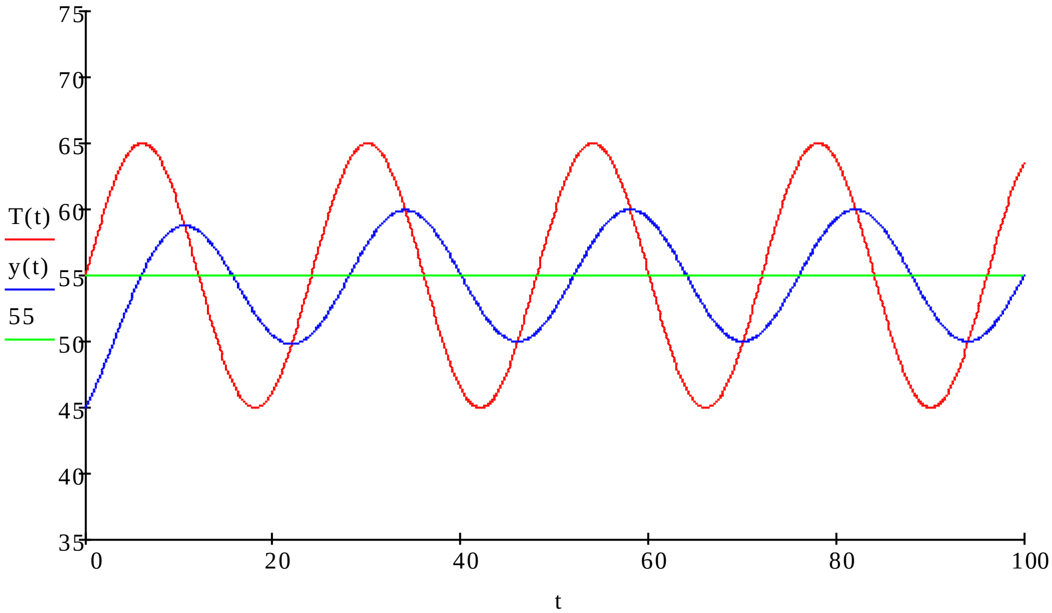
has solution(s)

$$-5.677 = C$$

$$y(t) := -4.323 \cos(0.262t) + 2.475 \sin(0.262t) + 55.0 - 5.677 e^{-.15 \cdot t}$$

See the Animation Newtons Law of Cooling---Note the maxima and minima correspond to the pts where y = T since these are precisely the pts where dy/dt = 0 . Note also how y follows T

$t := 0, .1 .. 100$



Using Laplace Transforms

$$\frac{dy}{dt} + .15y = 8.25 + 1.5\sin\left(\frac{\pi}{12}\cdot t\right)$$

$$(s + .15)\cdot F - 45 = \frac{8.25}{s} + \frac{0.393}{s^2 + 0.0685}$$

has solution(s)

$$F = \frac{\frac{8.25}{s} + \frac{0.393}{s^2 + 0.0685} + 45.0}{s + 0.15}$$

$$\frac{\frac{8.25}{s} + \frac{0.393}{s^2 + 0.0685} + 45.0}{s + 0.15}$$

has inverse Laplace transform

$$-4.32 \cdot \cos(0.262t) + 2.48 \cdot \sin(0.262t) + -5.68 \cdot e^{-0.15 \cdot t} + 55.($$

$$y(t) := -4.32 \cdot \cos(0.262t) + 2.48 \cdot \sin(0.262t) + -5.68 \cdot e^{-0.15 \cdot t} + 55.($$