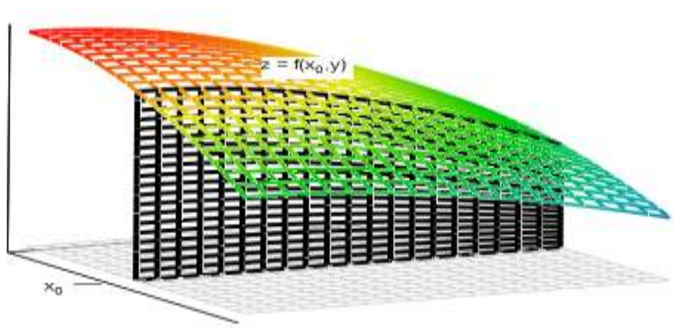
Double Integrals over Non-Rectangular Domains

Recall from our lecture on double integrals over rectangular domains we started by considering a cross-section for a fixed value of x. We computed the Area of this cross section using

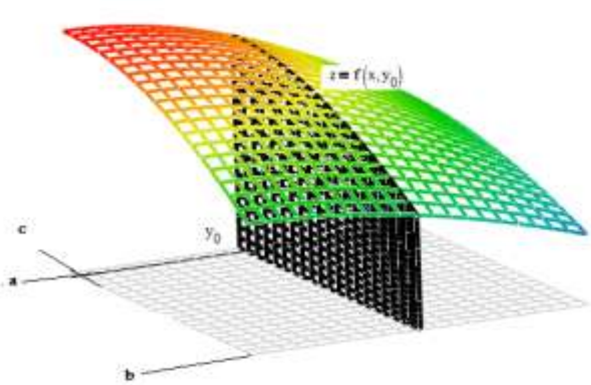
$$\int_c^d f(x,y)\,dy.$$



Since the volume is the integral with respect to x of the cross-sectional area-- $\int_a^b A(x)\,dx$ We obtained:

$$\int\int f(x,y)\,dA \quad = \quad \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Similarly we could start by fixing a value of y

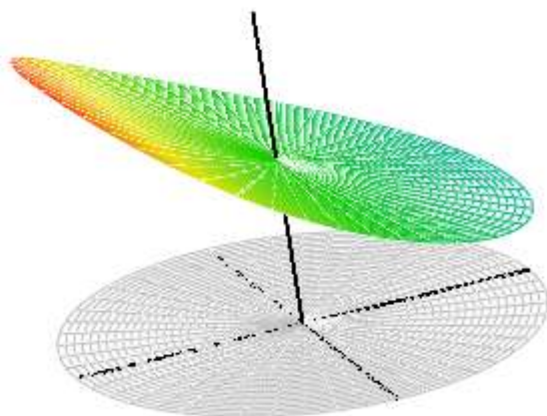


In which case we obtain

$$\int\int f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Suppose however we have a non-rectangular domain, for example suppose we have a circular domain.

In the graph below we have $x^2 + y^2 = 9$.



How do we now calculate $\int\int f(x,y)\,dA$?

The idea is basically the same as with the case of a rectangular domain.

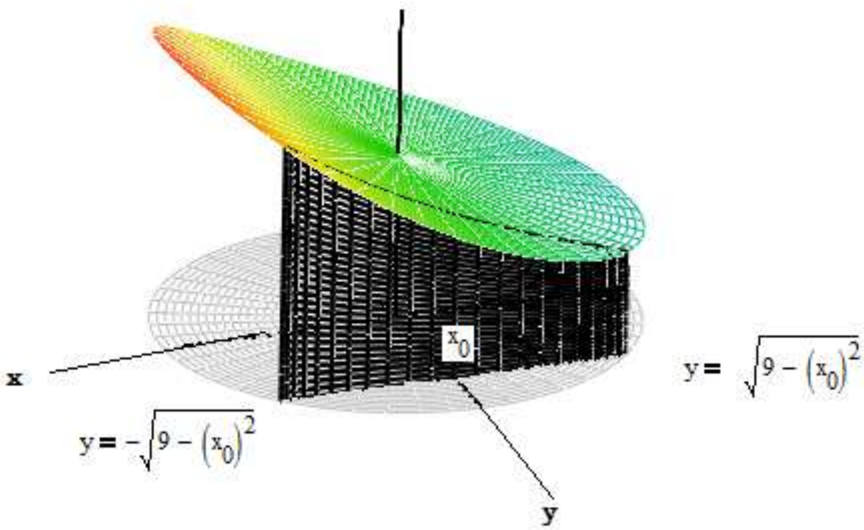
We start by fixing a value of x,computing the cross-sectional area by integrating with respect to y and then integrating this result with respect to x.

See Animation 11.

The main difference is that at each value of x the limits of integration change .

See Animation 10.

So let's consider then one cross section



Then at x_0 $$A(x_0) = \int_{-\sqrt{9-(x_0)^2}}^{\sqrt{9-(x_0)^2}} f(x_0, y)\, dy$$

Since this is true at every x we have in general

$$A(x) = \int_{-\sqrt{9-(x)^2}}^{\sqrt{9-(x)^2}} f(x, y)\, dy$$

Therefore since x varies from -3 to 3 :

$$\int\int f(x,y)\, dA = \int_{-3}^{3} A(x)\, dx = \int_{-3}^{3} \int_{-\sqrt{9-(x)^2}}^{\sqrt{9-(x)^2}} f(x, y)\, dy\, dx$$

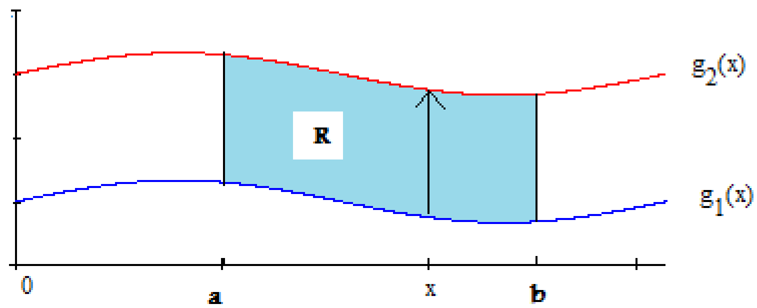
Be careful with double integrals over rectangular regions we could reverse the order of integration without thought. However over non rectangular regions it takes a little more work.

For example if we were to just ignorantly reverse the order of integration above our result would involve functions of x not a number--volumes and masses are numbers.

We will consider a few examples later but let's generalize the 2 types of basic regions we'll consider.

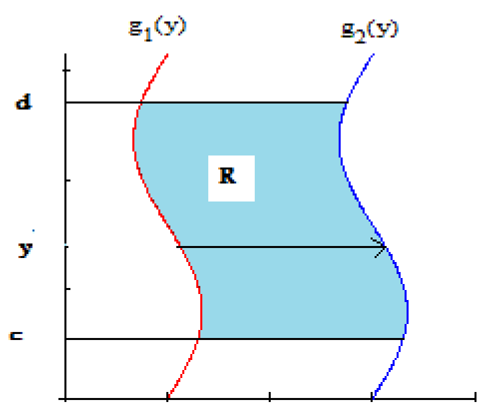
Generally we only graph the domain, denoted R, and not the surface z = f(x,y) to set up the double integral.

1. y varies between 2 functions of x and $a \leq x \leq b$.



$$\iint f(x,y)\,dA = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

1. x varies between 2 functions of y and $c \leq y \leq d$.



$$\int\int f(x,y)\,dA = \int_c^d A(y)\,dy = \int_c^d \int_{g_{1(y)}}^{g_{2(y)}} f(x,y)\,dx\,dy$$

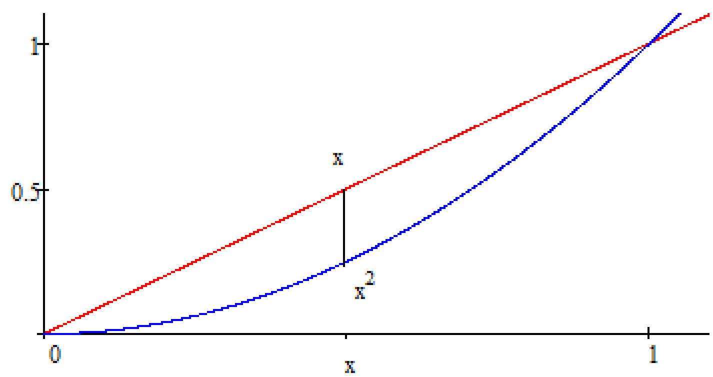
Example 1

Suppose R is the region bounded by y = x and y = x^2 . Suppose f(x,y) = $x^2\,y^3$.

a. Calculate the volume by first integrating with respect to y then x

b. Calculate the volume by first integrating with respect to x then y

a.



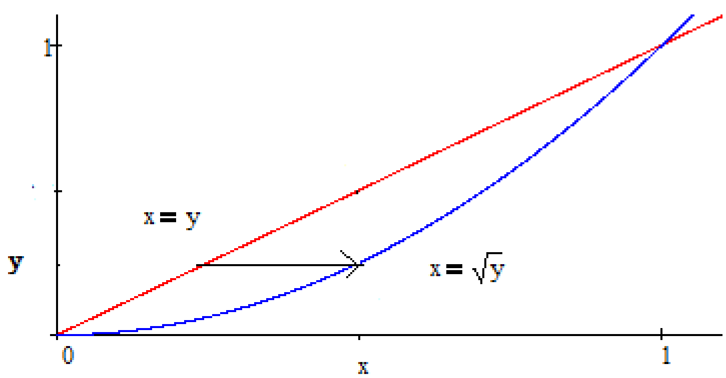
At each x, y varies from x^2 to x. We have $A(x) = \int_{x^2}^{x} x^2 \cdot y^3\,dy$

x varies from 0 to 1 Therefore we have:

$$\int\int f(x,y)\,dA = \int_0^1 \int_{x^2}^{x} x^2 \cdot y^3\,dy\,dx$$

$$\int_0^1 \int_{x^2}^{x} x^2 \cdot y^3 \, dy \, dx = \int_0^1 x^2 \cdot \left(\frac{y^4}{4} \cdot \Big|_{x^2}^{x} \right) dx = \frac{1}{4} \cdot \int_0^1 x^6 - x^{10} \, dx = \left[\left(\frac{x^7}{28} - \frac{x^{11}}{44} \right) \cdot \Big|_0^1 \right. = \frac{1}{28} - \frac{1}{44} = .013$$

b. Note here we have to invert our functions to get x as a function of y



$$\int \int f(x,y)\, dA \quad = \int_0^1 \int_y^{\sqrt{y}} x^2 \cdot y^3 \, dx dy$$

$$\int_0^1 \int_y^{\sqrt{y}} x^2 \cdot y^3 \, dx dy = \int_0^1 y^3 \cdot \frac{x^3}{3} \cdot \Big|_y^{\sqrt{y}} \, dx = \frac{1}{3} \cdot \int_0^1 y^{\frac{9}{2}} - y^6 \, dy = \left(\frac{2}{33} \cdot y^{\frac{11}{2}} - \frac{y^7}{21} \right) \cdot \Big|_0^1 = \frac{2}{33} - \frac{1}{21} = .01$$

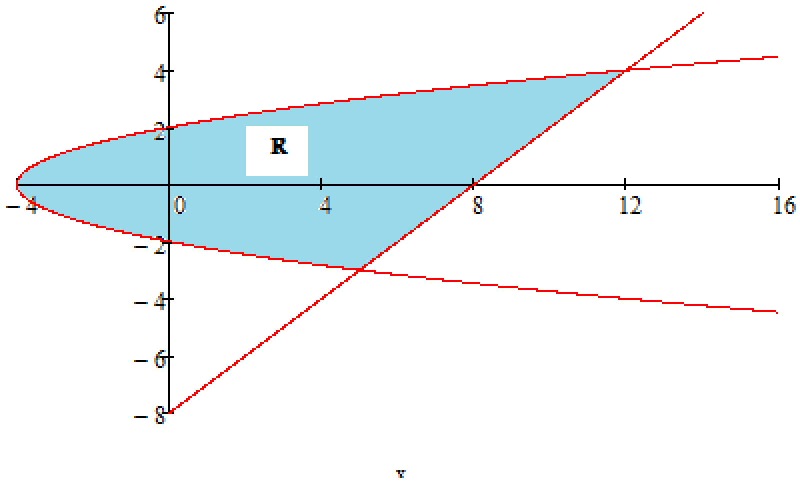
<u>Example 2</u>

Recall from our lecture on the double integral in terms of Riemann Sums if f(x,y) is the density then

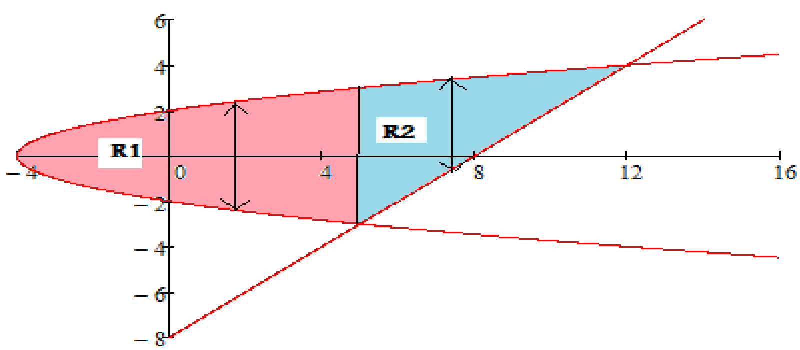
$\int \int f(x,y)\, dA$ calculates the volume of a plate occupied by R.

Suppose f(x,y) = $x^2 \cdot y^4$ is the density and R is the Region bounded by $x = y^2 - 4$ and $y = x - 8$.

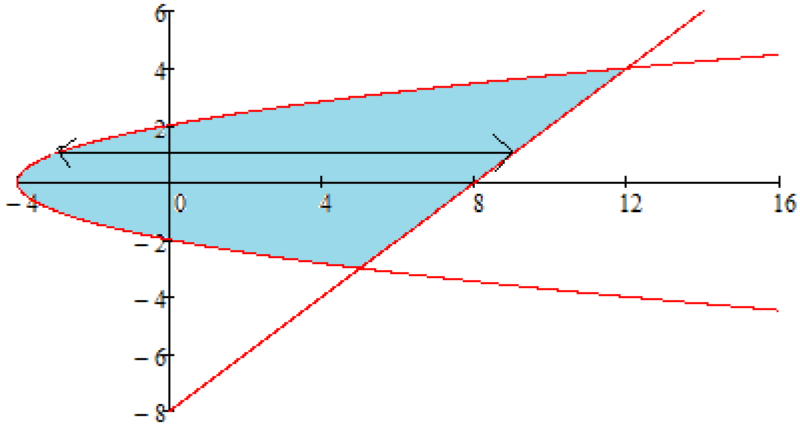
Find the mass .



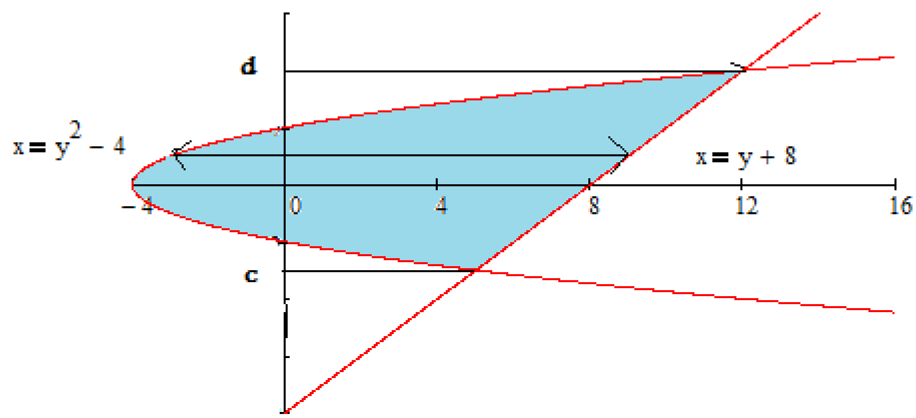
If we integrate first with respect to y and then with respect to x we need 2 integrals.



However if we integrate first with respect to x and then y we only need one integral.



So this is the approach we will take



$$\int\int f(x,y)\,dA = \int_{c}^{d}\int_{y^2-4}^{y+8} x^2 \cdot y^4\,dx\,dy$$

All we need then is c and d. These are simply the points of intersection of $x = y^2 - 4$ and $x = y + 8$

$y^2 - 4 = y + 8$

$y^2 - y - 12 = 0$

$(y - 4)\cdot(y + 3) = 0$

c = -3 and d = 4 we have

$$\int\int f(x,y)\,dA \quad = \int_{-3}^{4}\int_{y^2-4}^{y+8} x^2 \cdot y^4\,dx\,dy = 5.859\times 10^4$$

I've left the details of the integration to you -- honestly this is one you want the computer do .

The point being : the Computer can't set up the integral -- that's where your genius comes into play. But once we have the set up let the computer do the boring details.

Example 3.

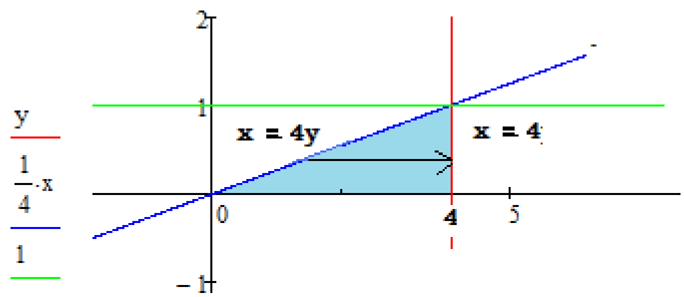
Evaluate $\displaystyle\int_{0}^{1}\int_{4y}^{4} e^{-x^2}\,dx\,dy$.

Well here neither we nor the computer can find the anti-derivative of e^{-x^2} . However sometimes by reversing the order of integration we can solve problems such as this. This doesn't mean simply switching the integrals.

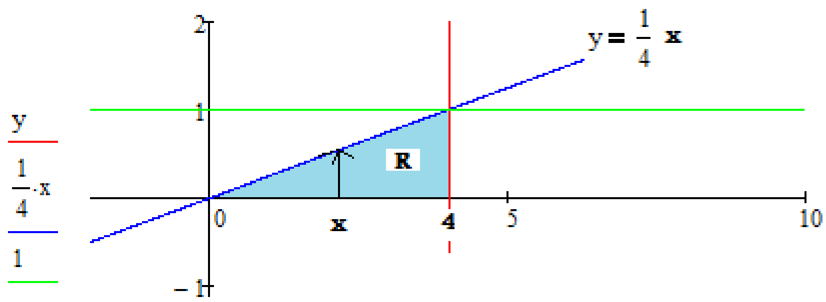
We start by graphing R. The curve x= 4 is a vertical line and x = 4y is the line y = 1/4x .

y= 0 is the x-axis and y = 1 is a horizontal line.

R is the triangular region in the graph below



Reversing the order of integration we have



Therefore $\int_0^1 \int_{4y}^4 e^{-x^2}\,dxdy = \int_0^4 \int_0^{\frac{1}{4}\cdot x} e^{-x^2}\,dxdy = \frac{1}{4}\cdot\int_0^4 x\cdot e^{-x^2}\,dx$ which we can now solve with the

u substitution $u = e^{-x^2}$.

$u = e^{-x^2}$

$u = -2\cdot x\cdot e^{-x^2}\cdot dx$

$\frac{1}{4}\cdot\int_0^4 x\cdot e^{-x^2}\,dx = \frac{-1}{8}\cdot\int_1^{e^{-16}} 1\,du = \frac{1}{8} - \frac{e^{-16}}{8} = .125$ to 3 figs.

Note : we took a problem which couldn't be solved by even Einstein (he's dead anyway) and by reversing the order of integration had a problem Einstein's dog could solve.