Double Integrals over Non-Rectangular Domains

Recall from our lecture on double integrals over rectangular domains we started by considering a cross-section for a fixed value of $x$. We computed the Area of this cross section using

$$\int_c^d f(x, y) \, dy.$$ 

Since the volume is the integral with respect to $x$ of the cross-sectional area--

$$\int_a^b A(x) \, dx$$

We obtained:

$$\int \int f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx$$

Similarly we could start by fixing a value of $y$
In which case we obtain

\[ \int \int f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy \]

Suppose however we have a non-rectangular domain, for example suppose we have a circular domain.

In the graph below we have \( x^2 + y^2 = 9 \).

How do we now calculate \( \int \int f(x, y) \, dA \)?

The idea is basically the same as with the case of a rectangular domain.

We start by fixing a value of \( x \), computing the cross-sectional area by integrating with respect to \( y \) and then integrating this result with respect to \( x \).

See Animation 11.

The main difference is that at each value of \( x \) the limits of integration change.

See Animation 10.
So let's consider then one cross section

Then at $x_0$, $A(x_0) = \int_{-\sqrt{9-(x_0)^2}}^{\sqrt{9-(x_0)^2}} f(x_0, y) dy$

Since this is true at every $x$ we have in general

$$A(x) = \int_{-\sqrt{9-(x)^2}}^{\sqrt{9-(x)^2}} f(x, y) dy$$

Therefore since $x$ varies from -3 to 3:

$$\int_{-3}^{3} \int_{-\sqrt{9-(x)^2}}^{\sqrt{9-(x)^2}} f(x, y) dy dx$$

Be careful with double integrals over rectangular regions we could reverse the order of integration without thought. However over non rectangular regions it takes a little more work.

For example if we were to just ignorantly reverse the order of integration above our result would involve functions of $x$ not a number—volumes and masses are numbers.
We will consider a few examples later but let's generalize the 2 types of basic regions we'll consider.

Generally we only graph the domain, denoted \( R \), and not the surface \( z = f(x,y) \) to set up the double integral.

1. \( y \) varies between 2 functions of \( x \) and \( a \leq x \leq b \).

\[
\int \int f(x,y) \, dA = \int_{a}^{b} A(x) \, dx = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx
\]

1. \( x \) varies between 2 functions of \( y \) and \( c \leq y \leq d \).
\[ \int \int f(x, y) \, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) \, dx \, dy \]

**Example 1**

Suppose \( R \) is the region bounded by \( y = x \) and \( y = x^2 \). Suppose \( f(x, y) = x^2 \cdot y^3 \).

a. Calculate the volume by first integrating with respect to \( y \) then \( x \)

b. Calculate the volume by first integrating with respect to \( x \) then \( y \)

a.

![Graph showing the region R bounded by y = x and y = x^2](image)

At each \( x \), \( y \) varies from \( \frac{x^2}{x} \) to \( x \). We have \( A(x) = \int_{x^2}^{x} x^2 \cdot y^3 \, dy \)

\( x \) varies from 0 to 1. Therefore we have:

\[ \int \int f(x, y) \, dA = \int_0^1 \int_{x^2}^{x} x^2 \cdot y^3 \, dy \, dx \]
\[
\int_0^1 \int_0^x x^2 \cdot y^3 \, dy \, dx = \int_0^1 \left( \frac{y^4}{4} \right) \, dx = \frac{1}{4} \int_0^1 x^6 - x^{10} \, dx = \left[ \frac{x^7}{28} - \frac{x^{11}}{44} \right]_0^1 = \frac{1}{28} - \frac{1}{44} = .013
\]

b. Note here we have to invert our functions to get \( x \) as a function of \( y \)

\[
\begin{align*}
\int \int f(x, y) \, dA &= \int_0^1 \int_y \sqrt{y} x^2 \cdot y^3 \, dx \, dy \\
\int_0^1 \sqrt{y} x^2 \cdot y^3 \, dx &= \int_0^1 \sqrt{y} x^2 \, dx = \frac{1}{3} \cdot \frac{9}{3} \cdot \int_0^1 x^2 - y^6 \, dy = \left( \frac{2}{33} \cdot \frac{11}{21} \cdot \frac{7}{21} \right) \cdot \left[ \frac{2}{33} - \frac{1}{21} \right] = .013
\end{align*}
\]

**Example 2**

Recall from our lecture on the double integral in terms of Riemann Sums if \( f(x, y) \) is the density then

\[
\int \int f(x, y) \, dA \quad \text{calculates the volume of a plate occupied by \( R \).}
\]
Suppose \( f(x,y) = x^2 \cdot y^4 \) is the density and \( R \) is the Region bounded by \( x = y^2 - 4 \) and \( y = x - 8 \).

Find the mass.

If we integrate first with respect to \( y \) and then with respect to \( x \) we need 2 integrals.

However if we integrate first with respect to \( x \) and then \( y \) we only need one integral.
So this is the approach we will take

All we need then is c and d. These are simply the points of intersection of $x = y^2 - 4$ and $x = y + 8$

\[
y^2 - 4 = y + 8
\]
\[
y^2 - y - 12 = 0
\]
\[
(y - 4) \cdot (y + 3) = 0
\]

$c = -3$ and $d = 4$ we have
\[ \int \int f(x,y) \, dA = \int_{-3}^{4} \int_{y^2-4}^{y^8} x^2 \cdot y^4 \, dxdy = 5.859 \times 10^4 \]

I've left the details of the integration to you -- honestly this is one you want the computer do.

The point being: the Computer can't set up the integral -- that's where your genius comes into play. But once we have the set up let the computer do the boring details.

**Example 3.**

Evaluate \[ \int_{0}^{1} \int_{4y}^{4} e^{-x^2} \, dx \, dy \, . \]

Well here neither we nor the computer can find the anti-derivative of \( e^{-x^2} \). However sometimes by reversing the order of integration we can solve problems such as this. This doesn't mean simply switching the integrals.

We start by graphing \( R \). The curve \( x = 4 \) is a vertical line and \( x = 4y \) is the line \( y = \frac{1}{4}x \).

\( y = 0 \) is the \( x \)-axis and \( y = 1 \) is a horizontal line.

\( R \) is the triangular region in the graph below

Reversing the order of integration we have
Therefore \[
\int_{0}^{1} \int_{0}^{4} e^{-x^2} \, dy \, dx = \int_{0}^{4} \int_{0}^{1} e^{-x^2} \, dy \, dx = \frac{1}{4} \int_{0}^{4} x \cdot e^{-x^2} \, dx \]
which we can now solve with the u substitution \[u = e^{-x^2}.
\]

\[u = e^{-x^2}
\]

\[u = -2 \cdot x \cdot e^{-x^2} \, dx
\]

\[\frac{1}{4} \cdot \int_{0}^{4} x \cdot e^{-x^2} \, dx = \frac{-1}{8} \cdot \int_{1}^{e^{16}} \, du = \frac{1}{8} - \frac{e^{-16}}{8} = .125 \quad \text{to 3 figs.}
\]

Note: we took a problem which couldn't be solved by even Einstein (he's dead anyway) and by reversing the order of integration had a problem Einstein's dog could solve.