# Line Integrals

In Physics 1 or Calculus 2 you were introduced to the basic idea of work.

If F is a vector field constant in both magnitude and direction and acts on an object as it moves along in a straight line in one dimension then Work = Force x Distance.

For example if a 5lb object is raised vertically 12 ft then W = 60 ft-lb of work.

Work is the change in the mechanical energy of a body i.e. the sum of potential and kinetic energy. In our example above the 60 ft-lb is the change in the potential energy of the object.

In Calculus 2 you allowed for F to change in magnitude but not direction--for example suppose

our object is a leaking bag whose weight could be 5 - 5/12x. Then the work becomes the

integral: $W = \int_a^b F(x)\,dx = \int_0^{12} \left(5 - \frac{5}{12}\cdot x\right) dx = 3($ ft lbs

But even this is much too restrictive.

Suppose $\vec{F} = f(t)\cdot\vec{i} + g(t)\vec{j}$ is any vector field (For now we'll assume $\vec{F}$ is a Force Field but

it could also represent a velocity field). Suppose $\overrightarrow{r(t)} = x(t)\cdot\vec{i} + y(t)\cdot\vec{j}$ is the parameterization

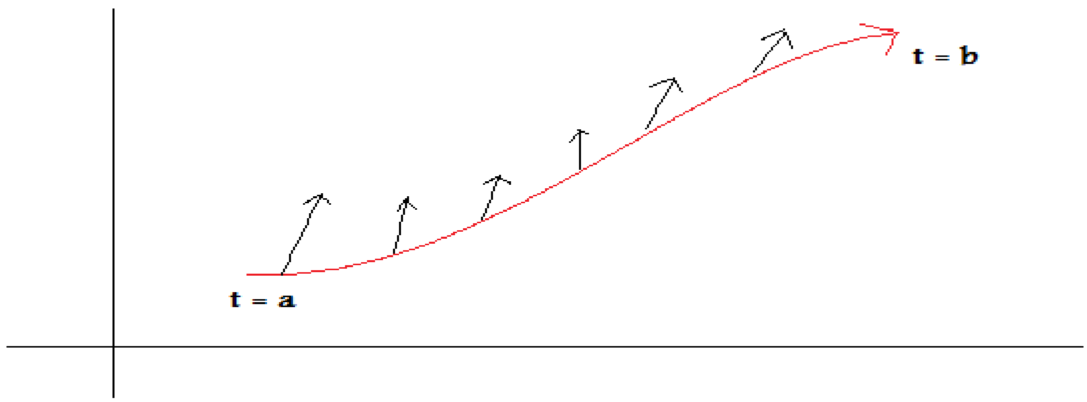
of any curve in 2-space. Then how do we calculate the work done in this more general case?

In fact $\vec{F}$ could be a vector field in 3-space and $\overrightarrow{r(t)}$ could be the parameterization of a curve in

3-space and again how would we calculate the work done?

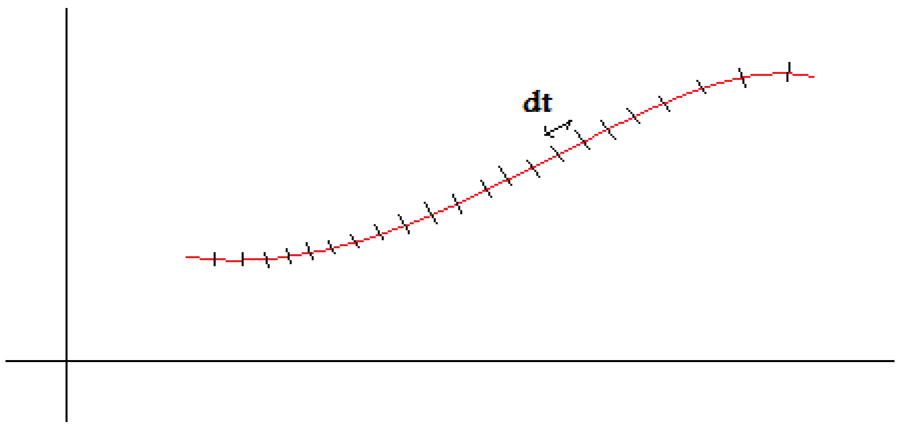
To get some idea of what we're talking about you may want to view the 6 animations now which illustrate the motion along different paths in different vector fields.

In the diagram below we have a curve in a vector field such that at each point $\vec{F}$ is different in both magnitude and direction. (We do assume the component functions are at least continuous).

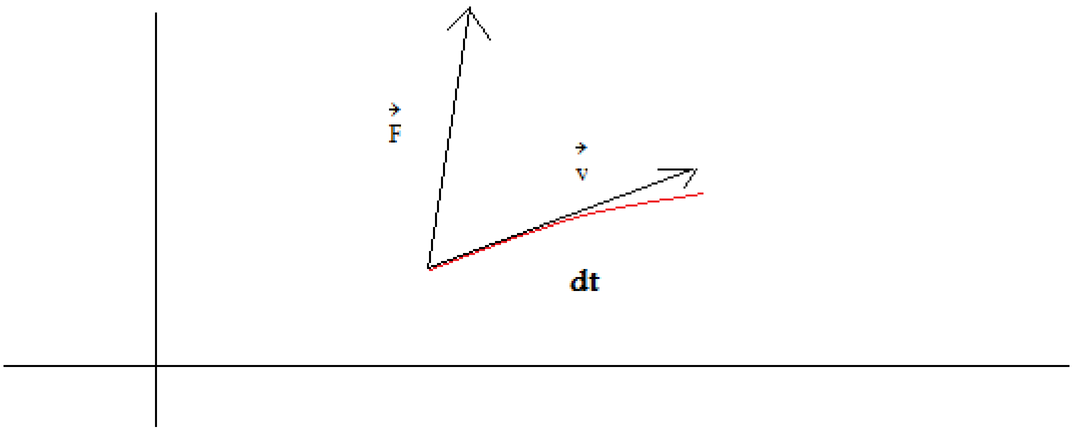



The direction of increasing parameter is up and to the right so that the particle is at its initial pt at t = a and is at its final pt at t = b.

As we so often do in Calculus we divide the trajectory into a large number of small segments which represent the motion over small time intervals dt.



We'll blow up our one small segment:




Then over dt $\vec{F}$ is constant and the direction of motion is in the direction of $\vec{v}$, the velocity vector. Over Δ t the speed is constant $||\vec{v}||$ so the distance traveled is $||\vec{v}||dt$

The component of the force in the direction of motion is $\left|\left|\text{proj}_{\vec{v}}\vec{F}\right|\right| = \dfrac{\vec{F}\cdot\vec{v}}{\left|\left|\vec{v}\right|\right|}$

Therefore the differential amount of work over time dt is dw = $\vec{F}\cdot\vec{v}$ dt.

It follows then the total work is $W = \int_a^b \vec{F}\cdot\vec{v}\,dt$ which we usually write $W = \int_a^b \vec{F}\cdot\overrightarrow{\left(\frac{dr}{dt}\right)}\,dt$ .

This is called the line integral of $\vec{F}$ along C where C is the curve with parameterization $\overrightarrow{[r(t)]}$ .

We also use the notation

$$\int_C f dx + g dy$$

Where f and g are the x and y component functions respectively of the vector field $\vec{F}$ or in 3 -space:

$$\int_C f\,dx + g\,dy + h\,dz$$

But in either case we use $W = \int_a^b \vec{F} \cdot \overrightarrow{\left(\frac{dr}{dt}\right)}\,dt$ to calculate the line integral in general.

Yet another notation in use is

$$W = \int_a^b \vec{F} \cdot \overrightarrow{dr}$$

If our vector field is a velocity field then the line integral calculates the circulation of the flow field along the curve C.