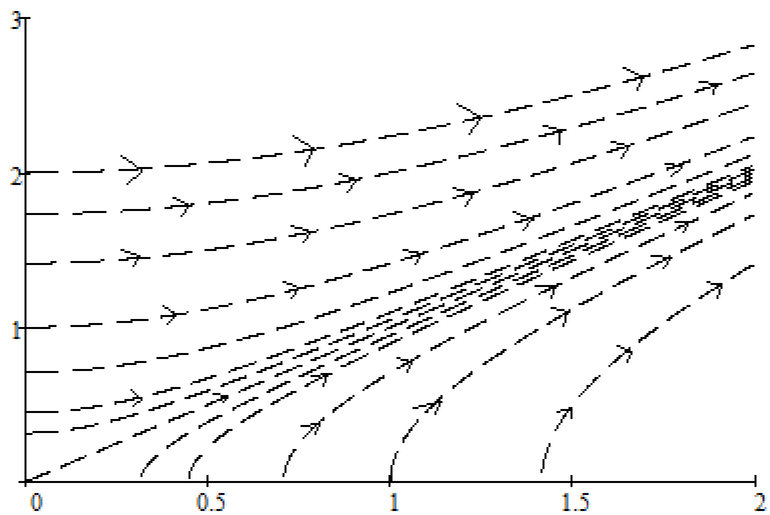
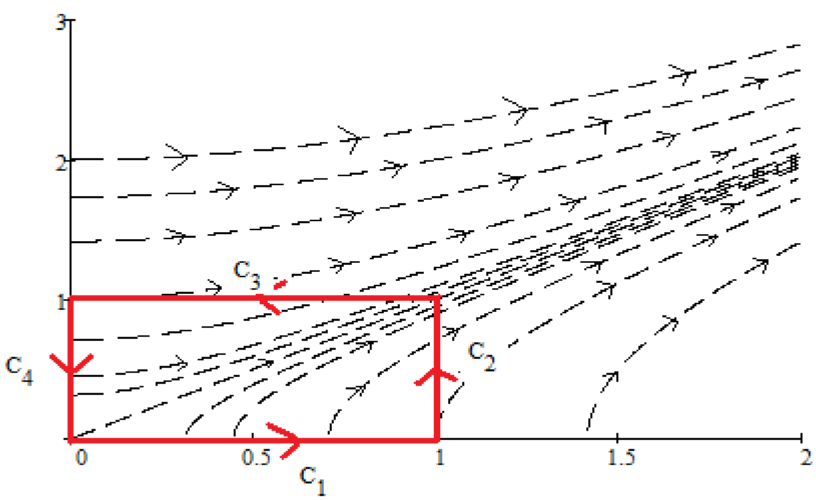
Solutions to Animations 4 - 6

The vector field is $\vec{F} = y \cdot \vec{i} + x \cdot \vec{j}$



1. Solution to Animation 4 Suppose we travel counter clockwise on the unit square.



The Curve C is piecewise smooth so we break it down into the 4 smooth pieces that make up C.

$$\int_C f\,dx + g\,dy = \int_{C_1} f\,dx + g\,dy + \int_{C_2} f\,dx + g\,dy + \int_{C_3} f\,dx + g\,dy + \int_{C_4} f\,dx + g\,dy$$

Along C_1 $\quad \overrightarrow{r(t)} = t\cdot\vec{i} \quad 0 \le t \le 1 \quad$ and $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j} = t\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = \vec{i}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = 0$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = 0$

Along C_2 $\quad \overrightarrow{r(t)} = \vec{i} + t\cdot\vec{j} \quad 0 \le t \le 1 \quad$ and $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j} = t\cdot\vec{i} + \vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = \vec{j}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = 1$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = \int_0^1 1\,dt = 1$

Along C_3 $\quad \overrightarrow{r(t)} = (1-t)\cdot\vec{i} + 1\cdot\vec{j} \quad 0 \le t \le 1 \quad$ and $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j} = 1\cdot\vec{i} + (1-t)\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = -\vec{i}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = -1$ Therefore $\int_0^1 -1\,dt = \int_0^1 1\,dt = -1$

The line integral is negative because the vector field is basically oppositely directed to the direction of motion. The meaning of the line integral being negative is that the energy is decreasing--one way to think of this running against the wind versus having the wind at your back.

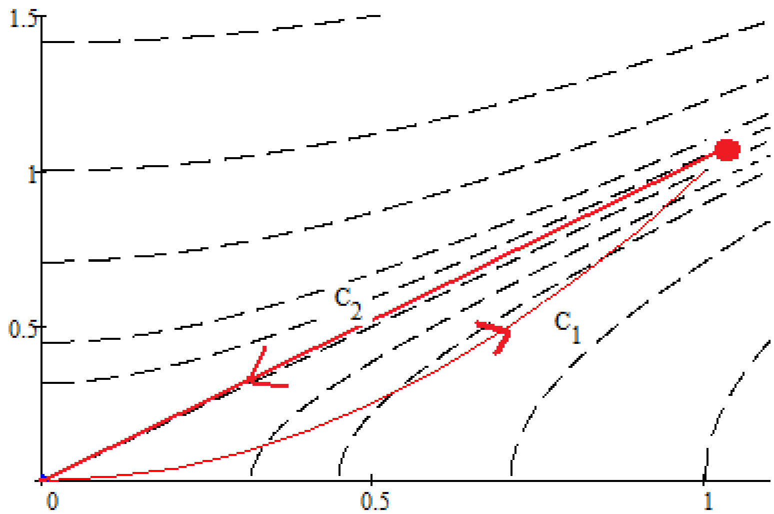
Along C_4 $\overrightarrow{r(t)} = (1-t)\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j} = (1-t)\cdot\vec{i}$

$\frac{d}{dt}\overrightarrow{r(t)} = -\vec{j}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = 0$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = 0$

Adding the 4 results we get:

$$\int_C f dx + g dy = 0$$

Solution to Animation 5 This time C is made up of 2 smooth pieces one along the parabola $y = x^2$ and $y = x$.



$$\int_C f dx + g dy = \int_{C_1} f dx + g dy + \int_{C_2} f dx + g dy$$

Along C_1 $\overrightarrow{r(t)} = t\cdot\vec{i} + t^2\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j} = t^2\cdot\vec{i} + t\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = \vec{i} + 2t\cdot j$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = 2t^2$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = \int_0^1 3t^2\,dt = 1$

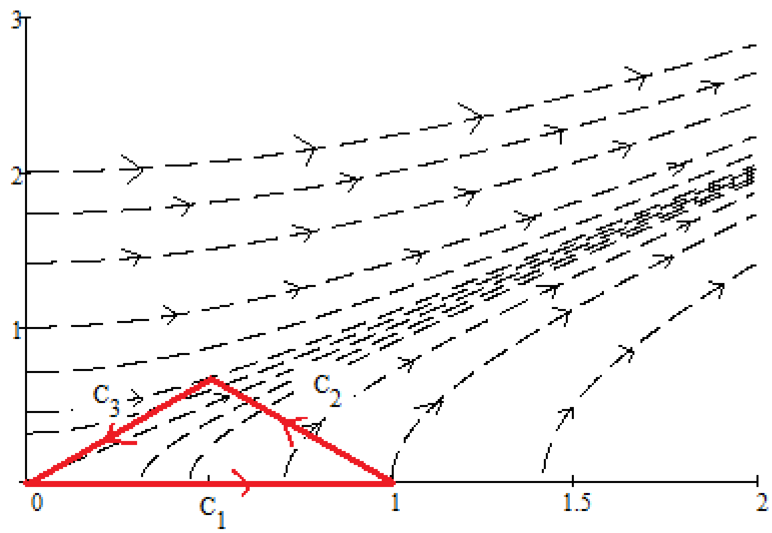
Along C_2 $\overrightarrow{r(t)} = (1-t)\cdot\vec{i} + (1-t)\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j} = (1-t)\cdot\vec{i} + (1-t)\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = -\vec{i} - \vec{j}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = -2 + 2\cdot t$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = \int_0^1 -2 + 2\cdot t\,dt = -1$

Adding the 2 results we get:

$$\int_C f dx + g dy = 0$$

Solution to Animation 6 This time C is made up of 3 smooth pieces connecting the vertices (0,0), (1,0) and (1/2,1/2)



$$\int_C f\,dx + g\,dy = \int_{C_1} f\,dx + g\,dy + \int_{C_2} f\,dx + g\,dy + \int_{C_3} f\,dx + g\,dy$$

Along C_1 as in the first example the Field and the direction of motion are perpendicular so the line integral is 0

Along C_2 $\overrightarrow{r(t)} = (1 - .5t)\cdot\vec{i} + .5t\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j} = .5t\cdot\vec{i} + (1 - .5t)\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = -.5\vec{i} + .5\vec{j}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = .5 - .5t$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = \int_0^1 .5 - .5t\,dt = .2!$

Along C_3 $\overrightarrow{r(t)} = (.5 - .5t)\cdot\vec{i} + (.5 - .5t)\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j} = (.5 - .5t)\cdot\vec{i} + (.5 - .5t)\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = -.5\cdot\vec{i} + -.5\cdot\vec{j}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = -.5 + .5\cdot t$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = \int_0^1 -.5 + .5\cdot t\,dt = -.25$

Adding the 3 results we get:

$$\int_C f\,dx + g\,dy = 0$$

It is no accident the 3 Line Integrals are 0. The vector Field is $\vec{F} = y\cdot\vec{i} + x\cdot\vec{j}$

This is an example of what is known as a Conservative vector field.

We'll explore this idea later but whenever we have a Conservative vector field the line integral is independent of path and depends only on the endpoints. When the initial and final pts are the same the line integral is 0.

In other words if the vector field is conservative the line integral around a closed path is 0.