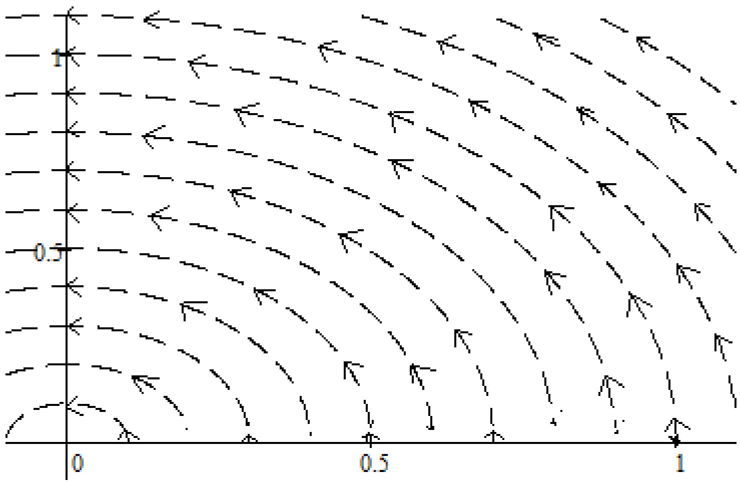
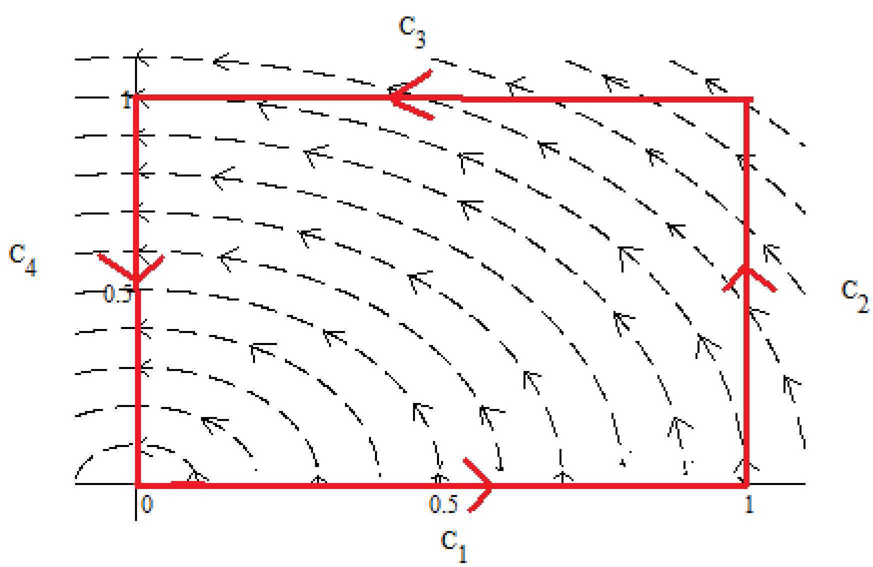
Solutions to Animations 1 - 3

In the 1st 3 animations the vector field is $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j}$



1. Solution to Animation 2 Suppose we travel counter clockwise on the unit square.



The Curve C is piecewise smooth so we break it down into the 4 smooth pieces that make up C.

$$\int_C f\,dx + g\,dy = \int_{C_1} f\,dx + g\,dy + \int_{C_2} f\,dx + g\,dy + \int_{C_3} f\,dx + g\,dy + \int_{C_4} f\,dx + g\,dy$$

Along C_1 $\overrightarrow{r(t)} = t\cdot\vec{i}$ $0 \le t \le 1$ and $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j} = t\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = \vec{i}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = 0$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = 0$

This makes sense since along C_1 the Force field is perpendicular to the direction of motion.

Along C_2 $\overrightarrow{r(t)} = \vec{i} + t\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j} = -t\cdot\vec{i} + \vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = \vec{j}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = 1$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = \int_0^1 1\,dt = 1$

The reason the line integral is positive is that along C_2 the Force field is basically in the same direction as the direction of motion.

Along C_3 $\overrightarrow{r(t)} = (1-t)\cdot\vec{i} + 1\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j} = -1\cdot\vec{i} + (1-t)\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = -\vec{i}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = 1$ Therefore $\int_0^1 1\,dt = \int_0^1 1\,dt = 1$

For the same reason as before the line integral is positive along C_3 because the Force field is basically in the same direction as the direction of motion.

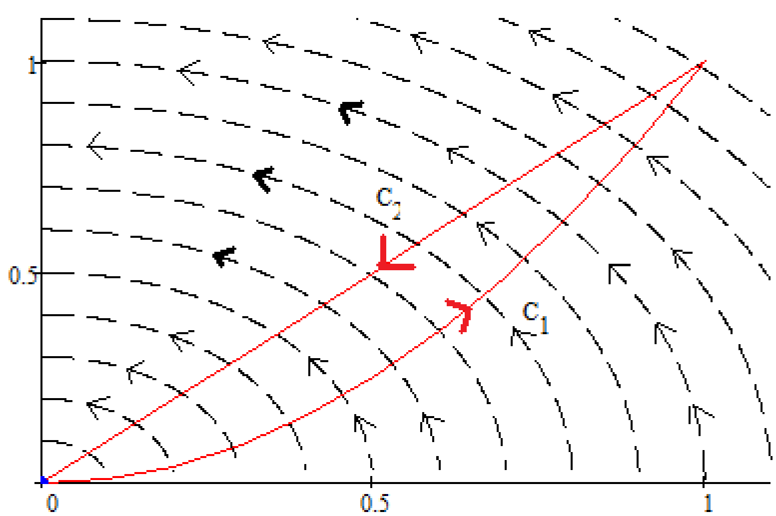
Along C_4 x =0 and the line integral will again be zero because the Force Field is $\vec{F} = -y \cdot \vec{i}$ and $\overrightarrow{r(t)} = (1-t) \cdot \vec{j}$

Therefore

$$\int_C f dx + g dy = 2$$

If you know Green's Theorem $\frac{\delta g}{dx} - \frac{\delta f}{dy} = 2$ and $\int_0^1 \int_0^1 \frac{\delta g}{dx} - \frac{\delta f}{dy} \, dx dy = 2$ If you don't know Green's Theorem you will soon be learning it.

<u>Solution to Animation 2</u> This time C is made up of 2 smooth pieces one along the parabola y = x^2 and y = x.



$$\int_C f dx + g dy = \int_{C_1} f dx + g dy + \int_{C_2} f dx + g dy$$

Along C_1 $\overrightarrow{r(t)} = t\cdot\vec{i} + t^2\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j} = -t^2\cdot\vec{i} + t\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = \vec{i} + 2t\cdot j$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = t^2$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$

Along C_2 $\overrightarrow{r(t)} = (1-t)\cdot\vec{i} + (1-t)\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j} = (t-1)\cdot\vec{i} + (1-t)\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = -\vec{i} - \vec{j}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = 0$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = 0$

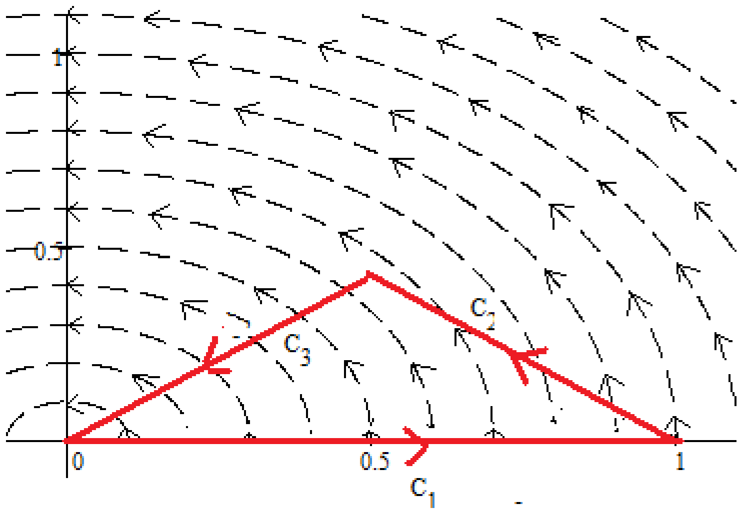
This makes sense since the flow lines are circles with orthogonal trajectories y = mx (see page on orthogonal trajectories)

Therefore

$$\int_C f dx + g dy = \mathbf{1/3}$$

Using Green's Theorem we obtain $\int_0^1 \int_{x^2}^{x} 2\,dy\,dx = \frac{1}{3}$

<u>Solution to Animation 3</u> This time C is made up of 3 smooth pieces connecting the vertices (0,0), (1,0) and (1/2,1/2)



$$\int_C fdx + gdy = \int_{C_1} fdx + gdy + \int_{C_2} fdx + gdy + \int_{C_3} fdx + gdy$$

Along C_1 as in the first example the Field and the direction of motion are perpendicular so the line integral is 0

Along C_2 $\overrightarrow{r(t)} = (1 - .5t)\cdot\vec{i} + .5t\cdot\vec{j}$ $\quad 0 \le t \le 1$ $\quad$ and $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j} = -.5t\cdot\vec{i} + (1 - .5t)\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = -.5\vec{i} + .5\vec{j}$ $\quad$ and $\quad F\cdot\frac{d}{dt}\overrightarrow{r(t)} = .5$ $\quad$ Therefore $\displaystyle\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = \int_0^1 .5dt = \frac{1}{2}$

Along C_3 $\overrightarrow{r(t)} = (.5 - .5t)\cdot\vec{i} + (.5 - .5t)\cdot\vec{j}$ $0 \le t \le 1$ and $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j} = -(.5 - .5t)\cdot\vec{i} + (.5 - .5t)\cdot\vec{j}$

$\frac{d}{dt}\overrightarrow{r(t)} = -.5\vec{i} + -.5\vec{j}$ and $F\cdot\frac{d}{dt}\overrightarrow{r(t)} = 0$ Therefore $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt = 0$ (Note again we are on y =x).

$$\int_C f dx + g dy = .5$$

Again let's consider the results from Green's Theorem $\int_0^{\frac{1}{2}}\int_y^{1-y} 2\,dx\,dy = 0.5$

You might be wondering why not just use Green's Theorem? Green's Theorem can only be used on a closed region. The form $\int_0^1 F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt$ can be used on any piecewise smooth curve.