Alternating Series

An alternating series is a series of the form $\sum_{k=1}^{n} \left[(-1)^{k+1} \cdot a_k\right]$ or $\sum_{k=0}^{n} \left[(-1)^{k} \cdot a_k\right]$

Thm Suppose we are given an alternating series $\sum_{k=1}^{n} \left[(-1)^{k+1} \cdot a_k\right]$ that satisfies the following conditions:

1. { a_k } is a decreasing sequence of positive terms i.e. a_k >0 and $a_1 > a_2 > a_3$ ..... $a_k > a_{k+1}$

2. $\lim_{k \to \infty} a_k = 0$

Then $\sum_{k=1}^{n} \left[(-1)^{k+1} \cdot a_k\right]$ converges

The proof involves 2 steps:

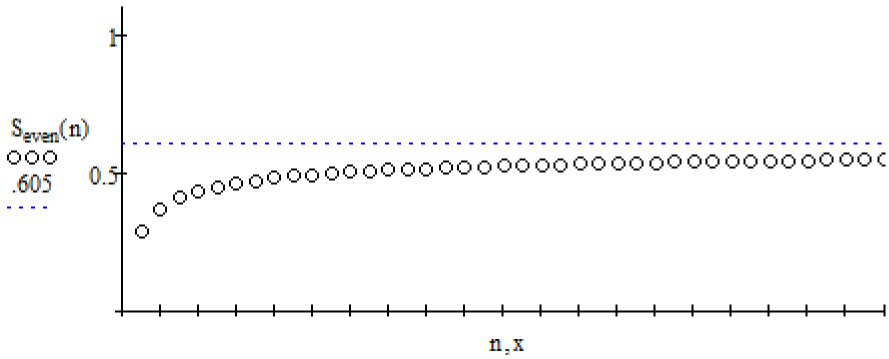
1. Show the subsequence of even partial sums is an increasing and bounded sequence.

2. Show the subsequence of odd partial sums converges to the same limit

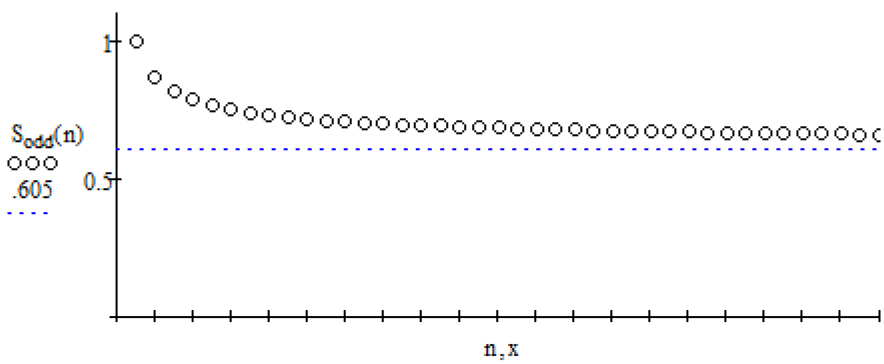
Therefore The sequence of partial sums converges.

Actually the graph and Animation1 and Animation 2 shows The subsequence of odd partial sums is a decreasing sequence bounded below, however we don't need this right now. We'll need this later however when we state and prove a second theorem on approximating the sum of an alternating series.

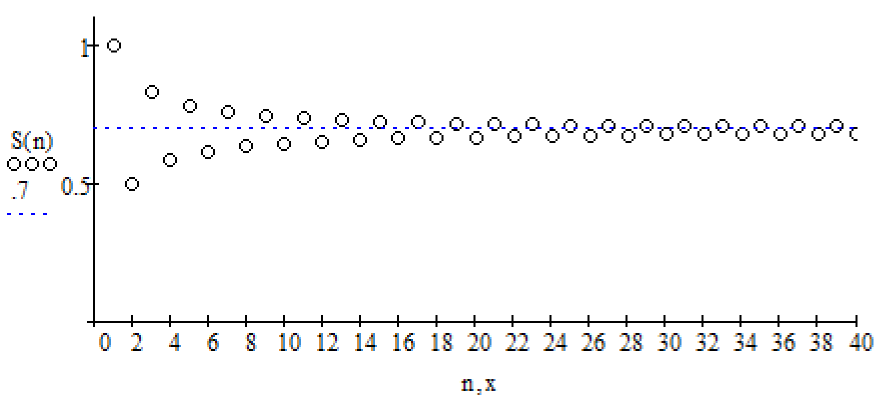
You might want to view animations 1 and animation 2 before continuing



1
0.5
$S_{even}(n)$
.605
n, x



1
0.5
$S_{odd}(n)$
.605
n, x



1
0.5
S(n)
.7
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
n, x

Proof Let n be even

1. Sn = $\sum_{k=1}^{n}\left[(-1)^{k+1}\cdot a_k\right] = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \text{ ....}+ a_{n-3} - a_{n-2} + a_{n-1} - a_n$

$= (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) \text{ ......}+ (a_{n-3} - a_{n-2}) + (a_{n-1} - a_n)$

$S_n = S_{n-2} + (a_{n-1} - a_n) > S_{n-2}$ since $a_{n-1} > a_n$

Therefore { S_n } is a increasing sequence

But also S_n can be written $a_1 - (a_2 - a_3) - (a_4 - a_5) - \text{.............}- (a_{n-2} - a_{n-1}) - a_n < a_1$

since $(a_{n-2} - a_{n-1})$ > 0 and since { a_n } is a decreasing sequence and a_n > 0 for every n.

Therefore { S_n } is bounded above by a_1.

Therefore { S_n } converges i.e. the subsequence of even partial sums converges. i.e. $\lim_{n\to\infty} S_n = L$.

2 Let n be given then we can write the sequence of even partial sums as S_{2n} and the sequence of odd partial sums as S_{2n+1}.

$S_{2n+1} - S_{2n} = S_{2n+1} = a_1 - a_2 + a_3 - \text{..........}- (a_{2n}) + a_{2n+1} - (a_1 - a_2 + a_3 - \text{..........}+ [(a_{2n-1}) + a_{2n}])$

$S_{2n+1} - S_{2n} = a_{2n+1}$

$\lim_{n\to\infty}(S_{2n+1} - S_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$

Therefore $\lim_{n \to \infty} \left(S_{2n+1} - S_{2n}\right) = 0 \qquad \lim_{n \to \infty} \left(S_{2n+1}\right) = \lim_{n \to \infty} \left(S_{2n}\right) = L$

Therefore The even and odd partial sums converge to the same limit and therefore $\lim_{n \to \infty} \left(S_n\right) = L$

which of course means $\sum_{k=1}^{n} \left[(-1)^{k+1} \cdot a_k\right]$ converges .

Theorem If $\sum_{k=1}^{n} \left[(-1)^{k+1} \cdot a_k\right]$ Converges then $\left|S - S_n\right| < a_{n+1}$ where S is the actual sum of the series. Before proving this we will do an example to demonstrate the power of this theorem.

What it is saying is that given any error bound we can approximate the actual sum of the series with its nth partial sum Sn simply by taking n large enough.

Example1 approximate $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ to within .001.

$$\left|S - S_n\right| < a_{n+1} = \frac{1}{\sqrt{n+1}} < .001$$

$$\frac{1}{\sqrt{n+1}} < .001$$

has solution(s)

$$999999.0 < n < \infty$$

So take n = 1,000,000

$$\sum_{k=1}^{10^6} \frac{(-1)^{k+1}}{\sqrt{k}} = 0.604$$

Example2 approximate $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ to within .001.

$$\left|S - S_n\right| < a_{n+1} = \frac{1}{(n+1)!} < .001$$

$$\frac{1}{(n+1)!} < .001$$

Here we will need to consider a table

$n := 0..10$

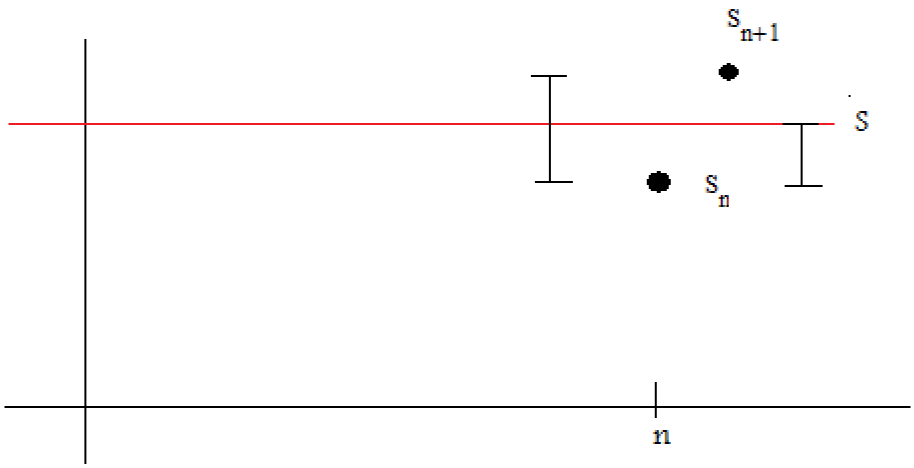
$n =$	$\frac{1}{(n+1)!} =$
0	1
1	0.5
2	0.167
3	0.042
4	$8.333 \cdot 10^{-3}$
5	$1.389 \cdot 10^{-3}$
6	$1.984 \cdot 10^{-4}$
7	$2.48 \cdot 10^{-5}$
8	$2.756 \cdot 10^{-6}$
9	$2.756 \cdot 10^{-7}$
10	$2.505 \cdot 10^{-8}$

So n + 1 > 6 so n > 5 So here we have convergence within .001 with just the 6th partial sum

$$\sum_{k=1}^{6} \frac{(-1)^{k+1}}{k!} = 0.632$$

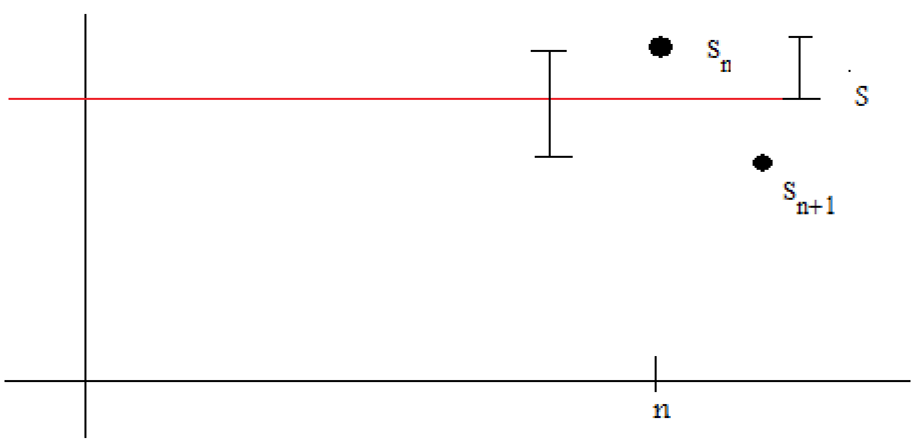
Proof

Case 1 Let n be even



$$\left|S - S_n\right| < \left|S_{n+1} - S_n\right| < \left|a_{n+1}\right| = a_{n+1}$$

Case 2 Let n be odd



$$\left|S_n - S\right| < \left|S_n - S_{n+1}\right| < \left|-a_{n+1}\right| = a_{n+1}$$