

Calculate the Laplace transform of $\cos(at - b)$ 3 ways.

1. Use Fundamental formulas for $\cos(at)$ and $\sin(at)$
2. Use the definition of Laplace transform
3. Use Mathcad

$$1. \cos(at - b) = \cos(at)\cos(b) + \sin(at)\sin(b)$$

$$L\{\cos(at - b)\} = \cos(b)L\{\cos(at)\} + \sin(b)L\{\sin(at)\}$$

$$L\{\cos(at - b)\} = \frac{\cos(b) \cdot s}{s^2 + a^2} + \frac{\sin(b) \cdot a}{s^2 + a^2}$$

$$2. L\{\cos(at - b)\} = \int_0^{\infty} e^{(-st)} \cdot \cos(at - b) dt$$

We'll start by considering the anti- derivative:

We integrate by parts:

$$\int e^{(-st)} \cdot \cos(at - b) dt$$

$$u = e^{-st}$$

$$dv = \cos(at - b)$$

$$du = -se^{-st}$$

$$v = \frac{-1}{a} \sin(at - b)$$

$$\int e^{(-st)} \cdot \cos(at - b) dt = \frac{-1}{a} e^{-st} \sin(at - b) - \frac{s}{a} \int e^{-st} \sin(at - b) dt$$

Now we integrate $\int e^{-st} \sin(at - b) dt$ by parts

$$u = e^{-st}$$

$$dv = \sin(at - b)$$

$$du = -se^{-st}$$

$$v = \frac{1}{a} \cos(at - b)$$

$$\int e^{-st} \sin(at - b) dt = \frac{1}{a} e^{-st} \cos(at - b) + \frac{s}{a} \int e^{(-st)} \cdot \cos(at - b) dt$$

$$\int e^{(-st)} \cdot \cos(at - b) dt = \frac{-1}{a} e^{-st} \sin(at - b) - \frac{s}{a} \int e^{-st} \sin(at - b) dt$$

$$\int e^{(-st)} \cdot \cos(at - b) dt = \frac{-1}{a} e^{-st} \sin(at - b) - \frac{s}{a^2} (e^{-st} \cos(at - b))$$

$$\left(1 + \frac{s^2}{a^2}\right) \int e^{(-st)} \cdot \cos(at - b) dt = \frac{-1}{a} e^{-st} \cdot \sin(at - b) - \frac{s}{a^2} (e^{-st} \cos(at - b))$$

$$\frac{a^2 + s^2}{a^2} \int e^{(-st)} \cdot \cos(at - b) dt = \frac{-1}{a} e^{-st} \cdot \sin(at - b) - \frac{s}{a^2} (e^{-st} \cos(at - b))$$

$$\int e^{(-st)} \cdot \cos(at - b) dt = \frac{-a}{s^2 + a^2} (e^{-st} \cdot \sin(at - b)) - \frac{s}{s^2 + a^2} (e^{-st} \cos(at - b))$$

Now evaluate from 0 to ∞

$$\lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{(-s \cdot t)} \cdot \cos(a \cdot t - b) dt = \lim_{\tau \rightarrow \infty} \left[\left(\frac{-s \cdot \cos(a \cdot t - b) \cdot e^{-s \cdot t} + a \cdot e^{-s \cdot t} \cdot \sin(b - a \cdot t)}{a^2 + s^2} \right) \cdot \Big|_0^{\tau} \right]$$

$$\lim_{\tau \rightarrow \infty} \left[\left(\frac{-s \cdot \cos(a \cdot t - b) \cdot e^{-s \cdot t} + a \cdot e^{-s \cdot t} \cdot \sin(b - a \cdot t)}{a^2 + s^2} \right) \cdot \Big|_0^{\tau} \right]$$

$$= \lim_{\tau \rightarrow \infty} \left(\frac{-s \cdot \cos(a \cdot t - b) \cdot e^{-s \cdot t} + a \cdot e^{-s \cdot t} \cdot \sin(b - a \cdot t)}{a^2 + s^2} \right) + \frac{s \cos(b) + a \sin(b)}{a^2 + s^2} = \frac{s \cos(b) + a \sin(b)}{a^2 + s^2}$$

3. Check by Letting Mathcad compute the Transform Directly:

$$\cos(a \cdot t - b) \quad \text{has Laplace transform} \quad \frac{a \cdot \sin(b) + s \cdot \cos(b)}{a^2 + s^2}$$