Lagrange Multipliers

In our last lecture we discussed a general method for finding global extrema for functions of 2 variables. When a region has a boundary with a smooth parameterization we can usually simplify the process with the following theorem.

Theorem (Lagrange's Theorem)

Let g(x,y) = c be a closed curve such that it has a smooth parameterization. Let f(x,y) be a function with continuous first partial derivatives defined on an open set containing g.

(In terms of constrained extrema problems f(x,y) is the objective function and g(x,y) = c is the constraint.)

Further suppose $\nabla g \neq 0$ at any point on this curve .

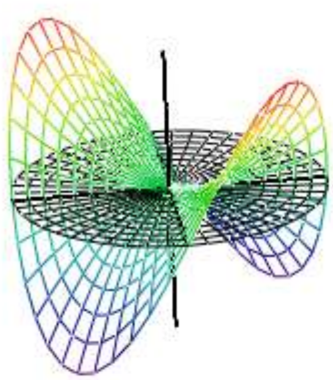
Then if f has an extremum at (x_0, y_0) then it occurs at a point where $\nabla f = \lambda \cdot \nabla g$.

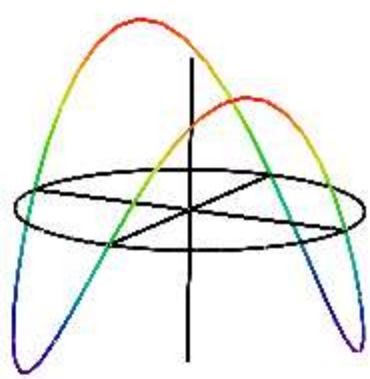
λ is a constant and is known as the Lagrange Multiplier.

We will prove this a little later but Let's consider an example first.

Example

Let $f(x,y) := x \cdot y$ on the unit circle g: $x^2 + y^2 = 1$. Find the maximum and minimum values .





$\nabla f = y \cdot \vec{i} + x \cdot \vec{j}$ $\nabla g = 2x \cdot \vec{i} + 2y \cdot \vec{j}$

$\nabla f = \lambda \cdot \nabla g$

$y \cdot \vec{i} + x \cdot \vec{j} = 2\lambda x \cdot \vec{i} + 2\lambda y \cdot \vec{j}$

Therefore we have :

$y = 2\lambda x$
$x = 2\lambda y$

Eliminating λ between the 2 equations we obtain:

$$\frac{y}{x} = \frac{x}{y}$$

$y^2 = x^2$ or $\boldsymbol{y = \pm x}$

using the constraint $x^2 + y^2 = 1$ we obtain:

$2 \cdot x^2 = 1$ so $x = \pm \frac{1}{\sqrt{2}}$.

We obtain the four critical points :

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 0.5$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = 0.5$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = -0.5$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -0.5$$

We have a maximum value of 1/2 which occurs at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ ***and*** $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

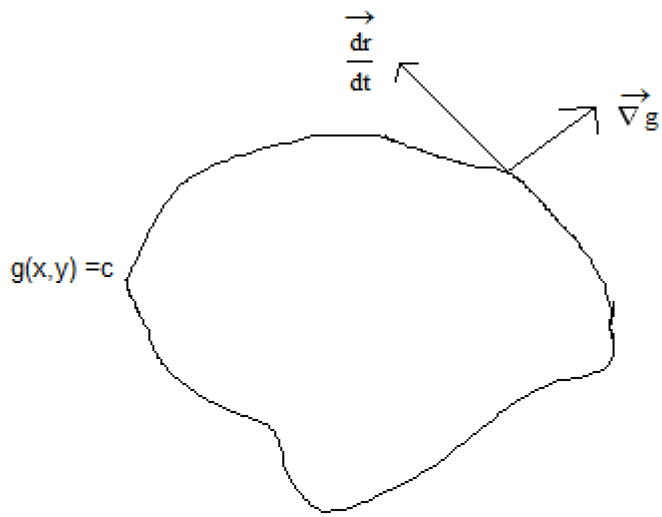
and a minimum value of -1/2 which occurs at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ ***and*** $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

Proof of Lagrange's Theorem

Let $\overrightarrow{r(t)} = x(t)\cdot\vec{i} + y(t)\cdot\vec{j}$ be a smooth parameterization of the constraint curve g(x,y) =c.

Then $\overrightarrow{\frac{dr}{dt}} = \frac{dx(t)}{dt}\cdot\vec{i} + \frac{dy(t)}{dt}\cdot\vec{j}$ is tangent to the curve at each point and ∇g is perpendicular to $\overrightarrow{\frac{dr}{dt}}$ at

each point since we can think of $\overrightarrow{r(t)}$ as a level curve of g(x,y).



We will show ∇f is perpendicular to $\frac{d\vec{r}}{dt}$ at an extremum and therefore parallel to ∇g which means ∇f is a scalar multiple of ∇g i.e. $\nabla f = \lambda \cdot \nabla g$

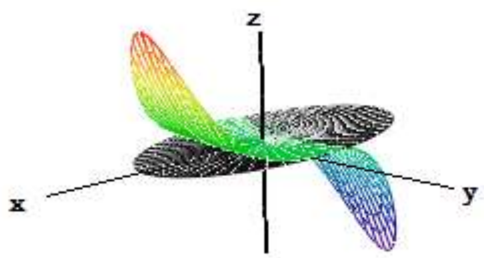
$$\nabla f = \frac{\delta f}{\delta x} \cdot \vec{i} + \frac{\delta f}{\delta y} \cdot \vec{j}$$

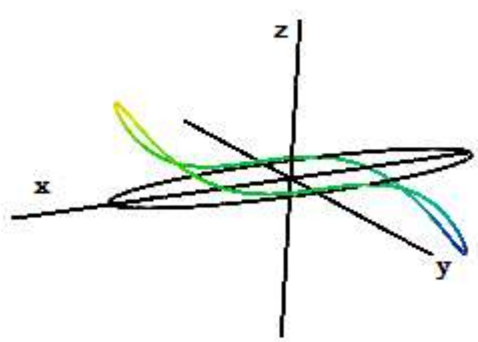
$$\nabla f \cdot \frac{d\vec{r}}{dt} = \left(\frac{\delta f}{\delta x} \cdot \vec{i} + \frac{\delta f}{\delta y} \cdot \vec{j} \right) \cdot \left(\frac{dx}{dt} \cdot \vec{i} + \frac{dy}{dt} \cdot \vec{j} \right) = \frac{\delta f}{\delta x} \cdot \frac{dx}{dt} + \frac{\delta f}{\delta y} \cdot \frac{dy}{dt} = \frac{df}{dt}$$

At an extremum $\frac{df}{dt} = 0$ and $\nabla f \cdot \frac{d\vec{r}}{dt} = 0$ i.e. ∇f is perpendicular to $\frac{d\vec{r}}{dt}$ and therefore parallel to ∇g .

Example 2

Suppose $f(x,y) = 4 \cdot x^3 + y^2$ Find the maximum and minimum values on the ellipse $x^2 + 2 \cdot y^2 = 1$





$\nabla f = 12 \cdot x^2 \cdot \vec{i} + 2y \cdot \vec{j}$ $\nabla g = 2x \cdot \vec{i} + 4y \cdot \vec{j}$

$\nabla f = \lambda \cdot \nabla g$

$12 \cdot x^2 \cdot \vec{i} + 2y \cdot \vec{j} = 2\lambda x \cdot \vec{i} + 4\lambda y \cdot \vec{j}$

Therefore we have :

$12x^2 = 2\lambda x$
$2y = 4\lambda y$

In equation 2 if $y \neq 0$ then λ = 2.

Using this in equation 1 we obtain $3 \cdot x^2 = x$ so x = 0 or 1/3

If x = 0 from the constraint $x^2 + 2 \cdot y^2 = 1$ $y = 1, -1$.

If x = 1/3 $\frac{1}{9} + 2 \cdot y^2 = 1$ $y = \frac{2}{3}, \frac{-2}{3}$

If $y = 0$ $x^2 = 1$ $x = 1, -1$

So we have 6 possible pts

$(0,1)$, $(0,-1)$, $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\left(\frac{1}{3}, \frac{-2}{3}\right)$, $(1,0)$ and $(-1,0)$

$f(x,y) := 4 \cdot x^3 + y^2$

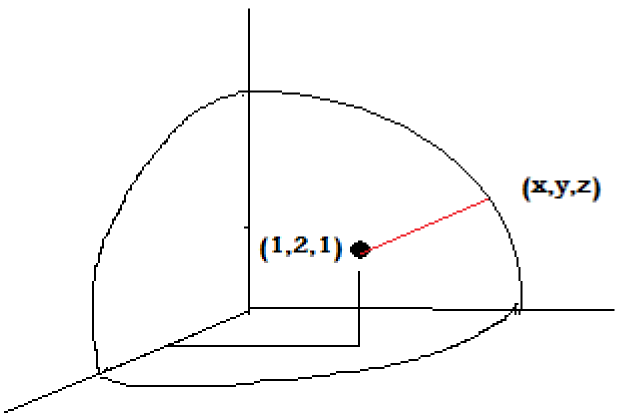
$f(0,1) = 1$ $f(0,-1) = 1$ $f\left(\frac{1}{3}, \frac{2}{3}\right) = 0.593$ $f\left(\frac{1}{3}, \frac{-2}{3}\right) = 0.593$ $f(1,0) = 4$ $f(-1,0) = -4$

We have a maximum value of 4 at (1,0) and a minimum value of -4 at (-1,0).

Example 3

We can also use Lagrange Multipliers for 3-D problems.

Find the points on the sphere $x^2 + y^2 + z^2 = 9$ closest to and furthest from the point (1,2,1) .



Let D be the square of the distance from (1,2,1) to any pt (x,y,z).

$$D = (x-1)^2 + (y-2)^2 + (z-1)^2$$

g: $x^2 + y^2 + z^2 = 9$

$$\nabla D = 2\cdot(x-1)\cdot\vec{i} + 2\cdot(y-2)\cdot\vec{j} + 2\cdot(z-1)\cdot\vec{k}$$

$$\nabla g = 2\cdot x\cdot\vec{i} + 2y\cdot\vec{j} + 2\cdot z\cdot\vec{k}$$

$$\nabla D = \lambda\cdot\nabla g$$

We obtain the 3 equations:

$\lambda x = x - 1$

$\lambda y = y - 2$

$\lambda z = z - 1$

Eliminating λ we obtain $\frac{x}{x-1} = \frac{y}{y-2} = \frac{z}{z-1}$

From the first 2 we obtain

$\frac{x}{x-1} = \frac{y}{y-2}$ which yields $xy - 2x = xy - y$ so y = 2x

Using the first and third

$\frac{x}{x-1} = \frac{z}{z-1}$ which yields $xz - x = xz - z$ so z = x

We use these results in the constraint $x^2 + y^2 + z^2 = 9$ to obtain $6 \cdot x^2 = 9$ $x = \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$

We have $\left(\sqrt{\frac{3}{2}}, 2 \cdot \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$ and $\left(-\sqrt{\frac{3}{2}}, -2 \cdot \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}\right)$

$d = \sqrt{D}$

$d(x,y,z) := \sqrt{(x-1)^2 + (y-2)^2 + (z-1)^2}$

$d\left(\sqrt{\frac{3}{2}}, 2 \cdot \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right) = 0.551$

$d\left(-\sqrt{\frac{3}{2}}, -2 \cdot \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}\right) = 5.449$

The minimum distance is 0.551 which occurs at $\left(\sqrt{\frac{3}{2}}, 2 \cdot \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$

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The maximum distance is 5.449 which occurs at $\left(-\sqrt{\frac{3}{2}}, -2 \cdot \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}\right)$