Prove the Identity $\int_0^x e^{(zx-z^2)}\,dz = e^{\frac{x^2}{4}}\cdot\int_0^x e^{\frac{-z^2}{4}}\,dz$

We star by defining:

$$I(x) = \int_0^x \exp\left(zx - z^2\right) dz$$

Completing the square we obtain:

$$zx - z^2 = -\left(z^2 - x\cdot z + \frac{x^2}{4}\right) + \frac{x^2}{4} = -\left(z - \frac{x}{2}\right)^2 + \frac{x^2}{4}$$

$$I(x) = \int_0^x \exp\left(zx - z^2\right) dz = e^{\frac{x^2}{4}}\cdot\int_0^x \exp\left[-\left(z - \frac{x}{2}\right)^2\right] dz$$

Using the product rule and the FTC

$$\frac{dI}{dx} = \frac{x}{2}\cdot\left[e^{\frac{x^2}{4}}\cdot\int_0^x \exp\left[-\left(z - \frac{x}{2}\right)^2\right] dz\right] + e^{\frac{x^2}{4}}\cdot e^{\frac{-x^2}{4}}$$

We obtain the linear first order DE

$$\frac{dI}{dx} = \frac{x}{2}\cdot I + 1$$

$$\frac{dI}{dx} - \frac{x}{2}\cdot I = 1$$

The Integrating factor is $e^{\frac{-x^2}{4}}$

$$\frac{d\left(I\cdot e^{\frac{-x^2}{4}}\right)}{dx} = e^{\frac{-x^2}{4}}$$

$$I\cdot e^{\frac{-x^2}{4}} = \int_0^x e^{\frac{-z^2}{4}}\,dz$$

$$I(x) = e^{\frac{x^2}{4}}\cdot\int_0^x e^{\frac{-z^2}{4}}\,dz = \int_0^x \exp\left(zx - z^2\right)dz$$