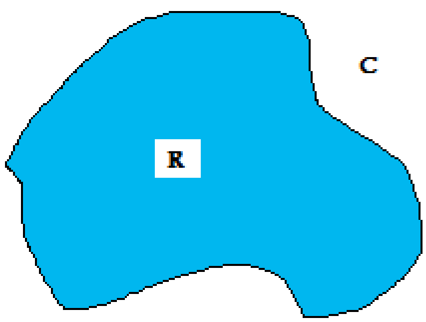
Green's Theorem

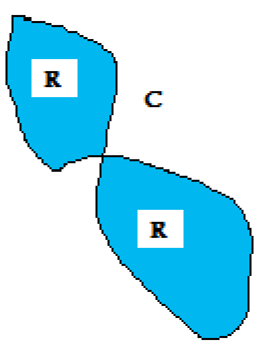
Simply Connected Regions

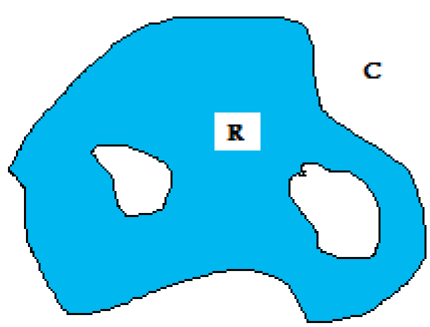
A region is called simply connected if its boundary is a single simple closed curve which does not intersect itself.



Below are examples of regions ***not*** simply connected. In the region on the left the curve

intersects itself. In the second The boundary consists of 3 simple curves (we call this multiple connected.)



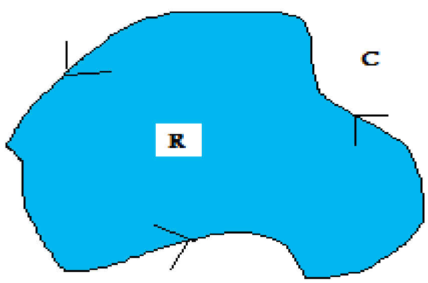


Green's Theorem:

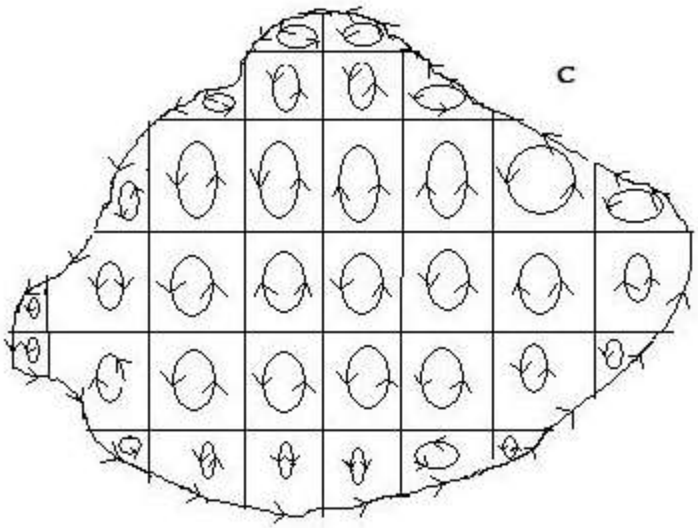
Let R be a simply connected region in 2-space bounded by a piecewise smooth curve C oriented counter clockwise.

Let $\vec{F} = f(x,y)\cdot\vec{i} + g(x,y)\cdot\vec{j}$ be such that f and g have continuous first partial derivatives on an open
set containing R.

Then $\int_C f dx + g dy = \iint_R (\partial g/\partial x - \partial f/\partial y) dA$.

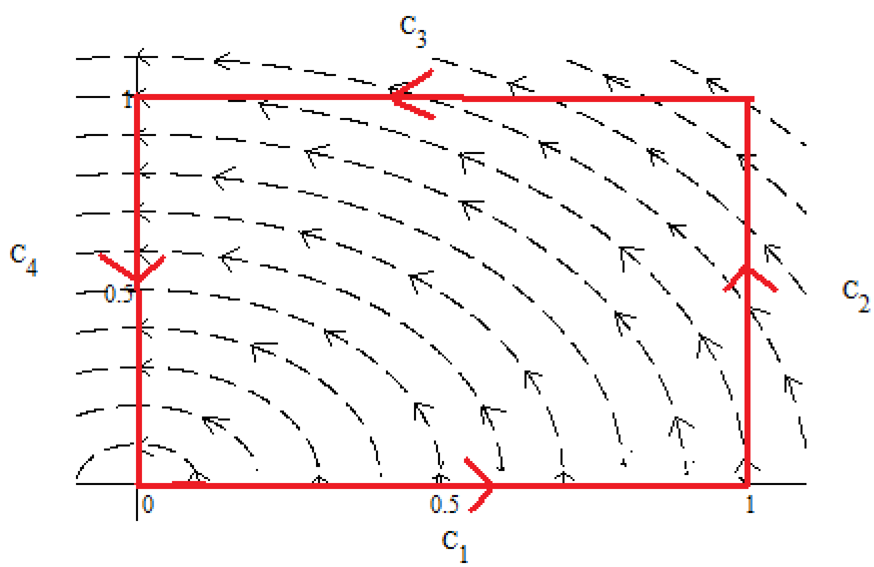


Instead of a formal proof we consider an intuitive proof. Recall from our discussion of the curl $\nabla \times \vec{F}$ in our discussion of path independence , $\partial g/\partial x - \partial f/\partial y$ represents the rotation of a vector field about a point . $\iint_R (\partial g/\partial x - \partial f/\partial y) dA$ is the sum of all the rotations throughout R. However if f and g have continuous first partial derivatives and if we partition R into a large number of rectangles then the circulations around each rectangle cancel except along the boundary giving the circulation of $\vec{F}$ along C . But the circulation of a vector field $\vec{F}$ along C is precisely what the line integral $\int_C f dx + g dy$ calculates . See Diagram below.



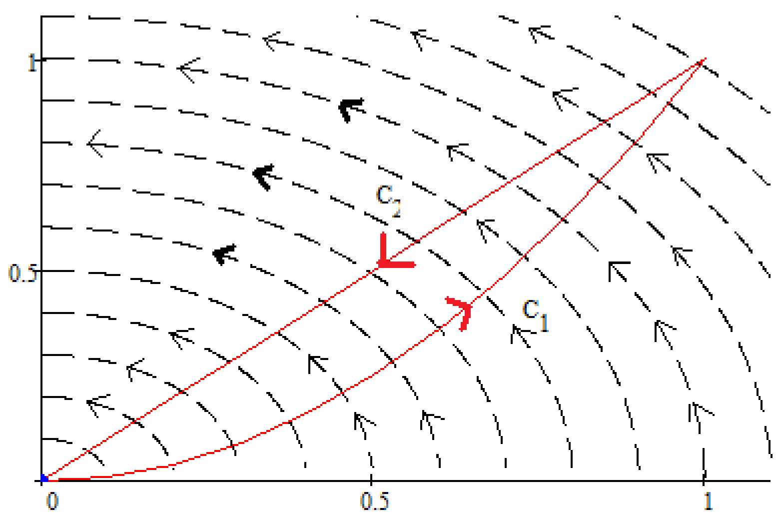
The following 3 examples correspond to the examples in the solutions to Animation's 1 - 3

1. Solution to Animation 1 Suppose we travel counter clockwise on the unit square. $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j}$



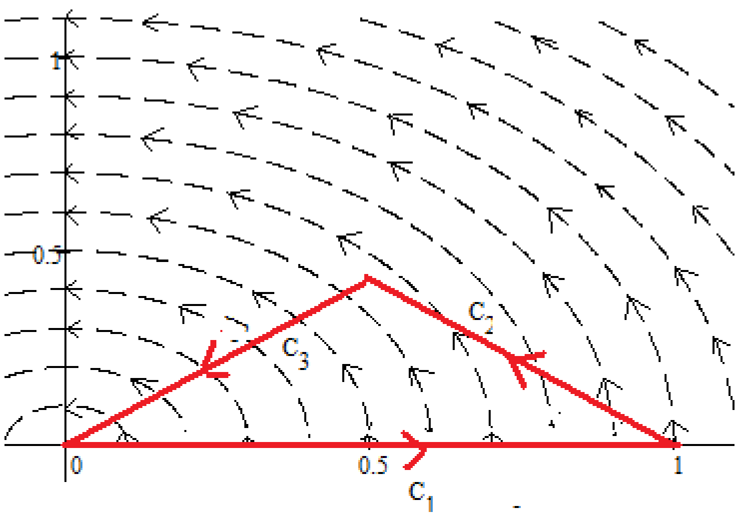
Since $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j}$ $\quad \frac{\delta g}{\delta x} - \frac{\delta f}{\delta y} = 2$ and $\int_0^1 \int_0^1 \frac{\delta g}{dx} - \frac{\delta f}{dy}\, dx\, dy = 2$

<u>Solution to Animation 2</u> This time C is made up of 2 smooth pieces one along the parabola $y = x^2$ and $y = x$.



Again Since $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j}$ $\quad \frac{\delta g}{\delta x} - \frac{\delta f}{\delta y} = 2$ we obtain $\int_0^1 \int_{x^2}^{x} 2\, dy\, dx = \frac{1}{3}$

Solution to Animation 3 This time C is made up of 3 smooth pieces connecting the vertices (0,0), (1,0) and (1/2,1/2) $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j}$ as before



Again Since $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j}$ $\frac{\delta g}{\delta x} - \frac{\delta f}{\delta y} = 2$ we obtain $\int_0^{\frac{1}{2}} \int_y^{1-y} 2\,dx\,dy = 0.5$

Example 4

Compute $\int_C \frac{-x^2\cdot y}{1+x^2}dx - \tan^{-1}(x)dy$ Where C is the unit square in example 1.

$\frac{\delta g}{\delta x} - \frac{\delta f}{\delta y} = \frac{1}{1+x^2} - \frac{-x^2}{1+x^2} = 1$ therefore $\int_C \frac{-x^2\cdot y}{1+x^2}dx - \tan^{-1}(x)dy = \int_0^1 \int_0^1 1\,dx\,dy = 1$

If you have some time compute this line integral using $\int_a^b F\cdot\frac{d}{dt}\overrightarrow{r(t)}\,dt$ on each of the four segments.

Example 5 Green's Theorem and Area

Suppose C encloses a region R whose area we want to determine.

Create the vector field $\vec{F} = -y\cdot\vec{j}$ (or $\vec{F} = x\cdot\vec{j}$)

Then $\int_C f dx + g dy = \iint_R (\partial g/\partial x - \partial f/\partial y) dA$. becomes $\int_C f dx + g dy = \iint_R dA = A_R$

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For example to compute the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the above result tells we can compute the line integral around the perimeter.

$\vec{F} = -y\cdot\vec{i} = -b\cdot\sin(t)\cdot\vec{i}$ $\quad \vec{r}(t) = a\cdot\cos(t) + b\cdot\sin(t)$ $\quad 0 \le t \le 2\pi$ $\quad \overrightarrow{\frac{dr}{dt}} = -a\cdot\sin(t)\cdot\vec{i} + b\cdot\cos(t)\cdot\vec{j}$

$$\vec{F}\cdot\overrightarrow{\frac{dr}{dt}} = -b\cdot\sin(t)\cdot\vec{i}\cdot\left(-a\cdot\sin(t)\cdot\vec{i} + b\cdot\cos(t)\cdot\vec{j}\right) = a\cdot b\cdot\sin^2(t)$$

$$A = \int_0^{2\cdot\pi} a\cdot b\cdot\sin^2(t)\,dt = a\cdot b\cdot\left(\frac{t}{2} - \frac{1}{2}\cdot\sin(t)\cdot\cos(t)\right)\cdot\Big|_0^{2\cdot\pi} = \pi\, a\cdot b$$

Compare to evaluating the integral to compute the same area using the traditional Calc 2 integral :

$$A = \int_{-a}^{a} b\cdot\sqrt{1 - \frac{x^2}{a^2}}\,dx$$ using trig substitution .