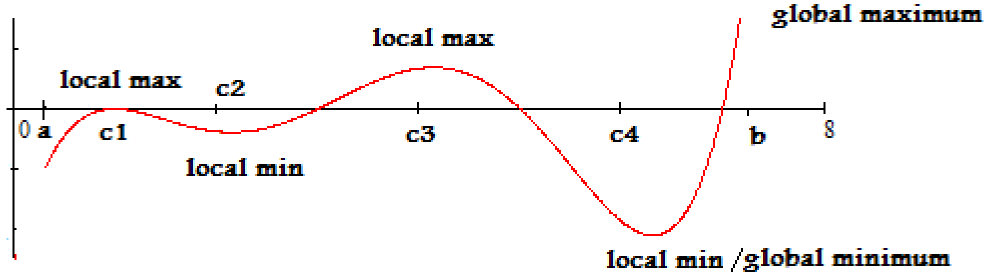
Global Extrema In General

We have previously defined the concept of local extrema for functions of 2 variables. Here we discuss global extrema for functions of 2 variables

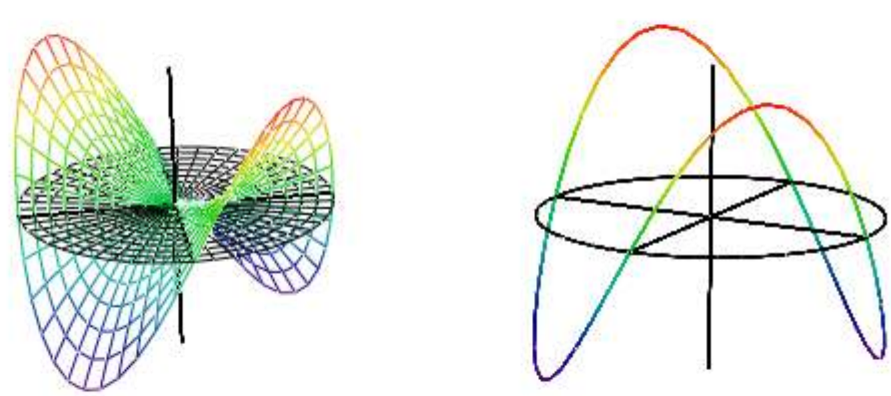
Recall that for functions of one variable that if f(x) is continuous on a closed and bounded interval [a,b] then f(x) has both a maximum and minimum value. We call these maximum and minimum values global extrema.



Further you learned in Calculus 1 that the global extrema occur either at local extrema or at the endpoints.

If you will we could refer to the enpoints as the boundary points.

For functions of 2 variables we have a similar situation : If a region is bounded by a closed curve then if f(x,y) is continuous it has both a maximum and minimum value which can occur either at a local extrema or on the boundary. In the graph below the extreme values occur on the boundary.



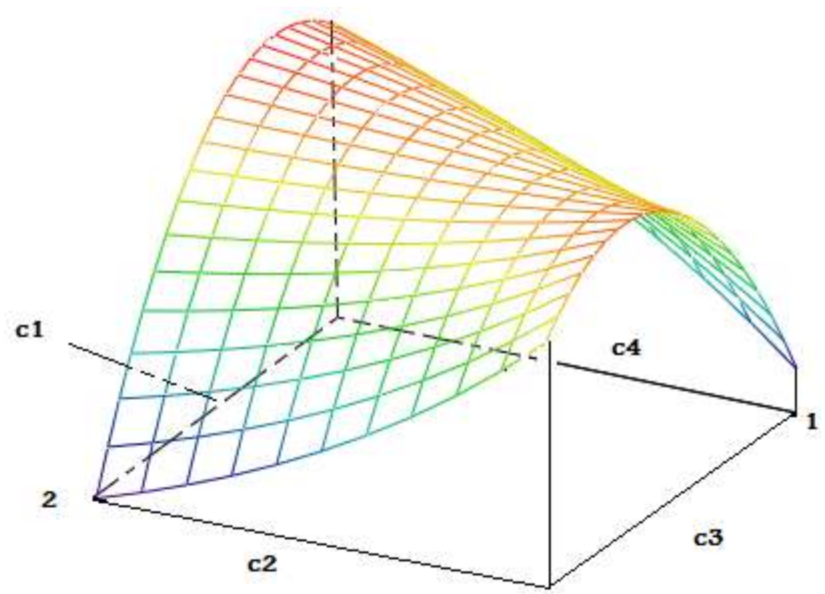
This is where the similarity ends. For functions of one variable we simply evaluated f(x) at the local extrema and at the boundary points to determine the maximum and minimum values.

However for functions of 2 variables the boundary is a curve and we can't simply evaluate f(x,y) at every point on the curve.

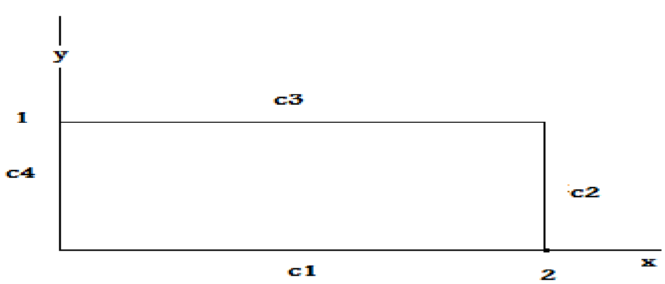
In this lecture we'll discuss how to solve this problem in general.

In the next lecture we introduce the method of Lagrange Multipliers which we generally use when the boundary does have a smooth parameterization.

Let $f(x,y) := x \cdot e^{y} - x^{2} - e^{y} + 3$ on the rectangle [0,2]x[0,1] Find the maximum and minimum values on the boundary.



We'll optimize f(x,y) on each of the 4 line segments which make up the boundary separately



On c1   y = 0 so $f(x,y) = x \cdot e^{y} - x^{2} - e^{y} + 3$ becomes $f(x,0) = x \cdot e^{0} - x^{2} - e^{0} + 3 = x - x^{2} + 2 = g(x)$
$0 \le x \le 2$.

Since we have a function of one variable we use the same technique as in Calculus 1

g(0) = 2   g(2) = 0

g ' (x) = 1 - 2x   so x = .5 is the only critical point   g(.5) = 2.5

On c1 the maximum value is 2.5 and the minimum value is 0

On c2   x = 2   so $f(x,y) = x \cdot e^{y} - x^{2} - e^{y} + 3$ becomes
$f(2,y) = 2 \cdot e^{y} - 2^{2} - e^{y} + 3 = e^{y} - 1 = g(y)$

$0 \le y \le 1$.

Again along c2 we have a function of one variable.

g(0) = 0   g(1) = e - 1 = 1.718

g ' (y) = e^{y} so there are no critical points

On c2 the maximum value is 1.718 and the minimum value is 0

On c3   y = 1   so $f(x,y) = x \cdot e^{y} - x^{2} - e^{y} + 3$ becomes $f(x,1) = x \cdot e - x^{2} - e + 3 = g(x)$

$0 \le x \le 2$.

g(0) = - e + 3 = .282   g(2) = e - 1 = 1.718

g ' (x) = e - 2x   we have a critical point at x = e/2   g(e/2) = 2.129

On c3 the maximum value is 2.129 and the minimum value is .282

Finally   on c4 x = 0   so $f(x,y) = x \cdot e^{y} - x^{2} - e^{y} + 3$ becomes $f(0,y) = -e^{y} + 3 = g(y)$

$0 \le y \le 1$

g(0) = 2   g(1) = -e + 3 = 2.129

g ' (y) = $-e^{y}$ there are no critical points.

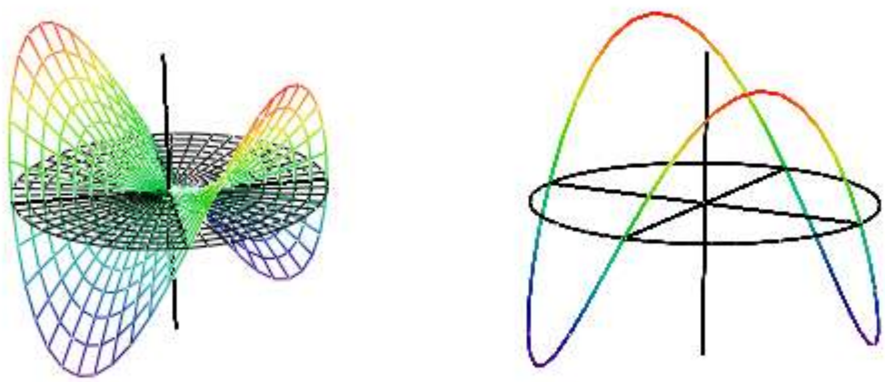
On c4 the maximum value is 2.129 and the minimum value is 2

Comparing all the results the maximum value is 2.5 and the minimum value is 0.

You may wonder if there is an easier way. If we have a smooth boundary we can use Lagrange Multipliers which we discuss in our next lecture.

Example 2

Let f(x,y) = xy on the unit disk $x^2 + y^2 \le 1$. Find the maximum and minmum values .

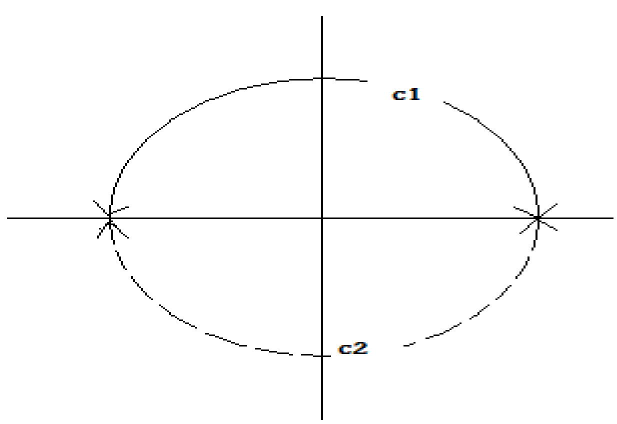


1. We start by looking for local extrema on the interior

$\nabla f = y \cdot \vec{i} + x \cdot \vec{j}$ The critical point is (0,0)   and f(0,0) = 0

2. We now look for the maximum and minimum values on the boundary.

We break the domain into the upper half of the unti circle - c1- and the lower half -c2.



On c1 $y = \sqrt{1-x^2}$ so $f(x,y) = x\cdot\sqrt{1-x^2} = g(x)$ $0 \le x \le 1$

g(0) = g(1) = 0

g ' (x) = $\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = -\frac{2\cdot x^2 - 1}{\sqrt{1-x^2}}$ so we have critical points at $x = \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

$g(\frac{1}{\sqrt{2}})$ = 1/2 $g(\frac{-1}{\sqrt{2}})$ = -1/2

On c1 the maximum value is 1/2 which occurs at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ ***and the minimum value is -1/2 which occurs at*** $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ ***.***

On c2 $y = -\sqrt{1-x^2}$ so $f(x,y) = -x\cdot\sqrt{1-x^2} = g(x)$ $0 \le x \le 1$

g(0) = g(1) = 0

g ' (x) = $-\sqrt{1-x^2} - \frac{-x^2}{\sqrt{1-x^2}} = -\frac{-2\cdot x^2 + 1}{\sqrt{1-x^2}}$ so we have critical points at $x = \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

$g(\frac{1}{\sqrt{2}})$ = -1/2 $g(\frac{-1}{\sqrt{2}})$ = 1/2

On c1 the maximum value is 1/2 which occurs at the point $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ ***and the minimum value is -1/2 which occurs at*** $(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ ***.***

Comparing the results of our local extrema and the extrema on the boundary

We have a maximum value of 1/2 which occurs at $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ ***and*** $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$

and a minimum value of -1/2 which occurs at $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ ***and*** $(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$

In our next lecture we'll solve this problem using Lagarange Multipliers.