Flux over a Parameterized Surfaces

We saw in the lab on parametric surface plots we could generate a cylinder with radius 1 and height 2 with the equations:

$x(t) = \cos(t)$ $y(t) = \sin(t)$ $z(s) = s$ $0 \le t \le 2\pi$ and $0 \le s \le 2$

We can also write this in the vector form:

$$\overrightarrow{r(s,t)} = \cos(t)\cdot\vec{i} + \sin(t)\cdot\vec{j} + s\cdot\vec{k}$$

In fact any parametric surface plot can be written in the form:

$$\overrightarrow{r(s,t)} = f_1(s,t)\cdot\vec{i} + f_2(s,t)\cdot\vec{j} + f_3(s,t)\cdot\vec{k}$$

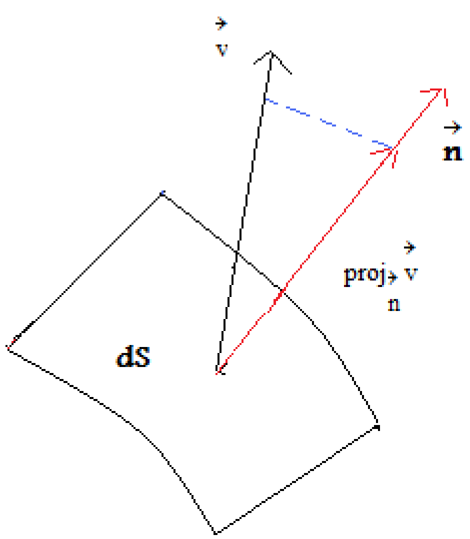
What we want to develop in this lecture is how do we compute the flux over a parameterized surface?

We start in the same way we did we developed the basic form of a flux integral:

We partition the surface into a large number of rectangular patches with area ds. On each of these patches $\vec{v}$ is approximately constant. We then project $\vec{v}$ onto $\vec{n}$ the unit normal.

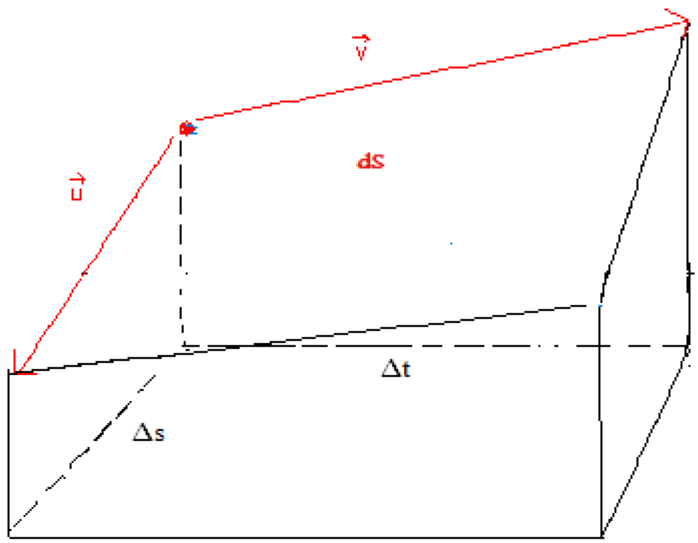
We then have the situation where we have a flow perpendicular to a flat surface therefore

$$d\Phi = \left\|\mathrm{proj}_{\vec{n}}\vec{v}\right\|\cdot ds = \vec{v}.\vec{n}\cdot ds$$



Where S is a surface in our parameter space

$\vec{u} = \vec{r}(s + \Delta s, t) - \vec{r}(s,t)$ $\qquad$ $\vec{v} = \vec{r}(s, t + \Delta t) - \vec{r}(s,t)$



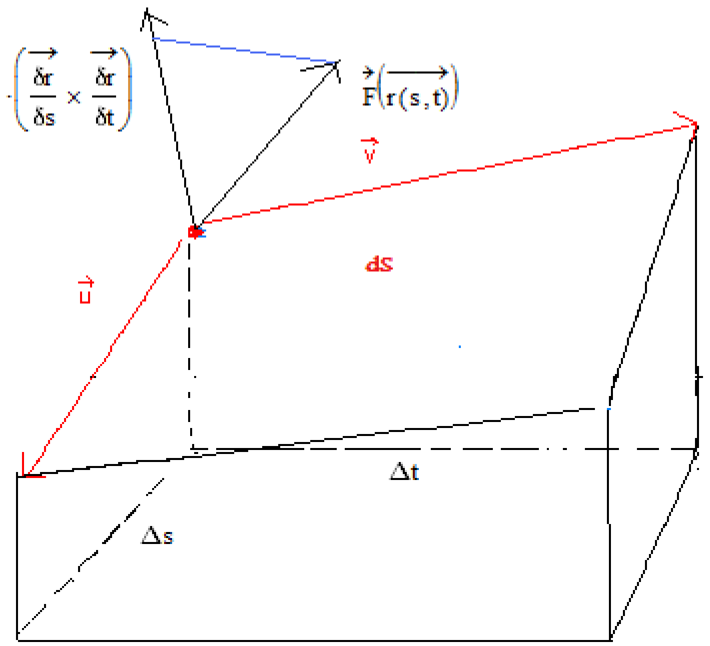
$$\vec{u} = \frac{\vec{r}(s + \Delta s, t) - \vec{r}(s,t)}{\Delta s} \cdot \Delta s \qquad \vec{v} = \frac{\vec{r}(s, t + \Delta t) - \vec{r}(s,t)}{\Delta t} \cdot \Delta t$$

Which gives us taking appropriate limits

$$\vec{u} = \overrightarrow{\frac{\partial r}{\partial s}} \cdot ds \qquad v = \overrightarrow{\frac{\partial r}{\partial t}} \cdot dt$$

The normal to this surface is then: $\overrightarrow{\frac{\partial r}{\partial s}} \times \overrightarrow{\frac{\partial r}{\partial t}}$ and the component of the Force parallel

to the normal is: $\left(\overrightarrow{projF}\right)_{\overrightarrow{\frac{\partial r}{\partial s}} \times \overrightarrow{\frac{\partial r}{\partial t}}} = \vec{F}\left(\overrightarrow{r(s,t)}\right) \cdot \frac{\left(\overrightarrow{\frac{\partial r}{\partial s}} \times \overrightarrow{\frac{\partial r}{\partial t}}\right)}{\left\|\left(\overrightarrow{\frac{\partial r}{\partial s}} \times \overrightarrow{\frac{\partial r}{\partial t}}\right)\right\|}$



The Area dS = $\left\|\vec{u} \times \vec{v}\right\| = \left\|\left(\vec{\frac{\delta r}{\delta s}} \cdot ds\right) \times \left(\vec{\frac{\delta r}{\delta t}} \cdot dt\right)\right\| = \left\|\vec{\frac{\delta r}{\delta s}} \times \vec{\frac{\delta r}{\delta t}}\right\| \cdot dsdt$

$$d\Phi = \vec{F}\left(\vec{r(s,t)}\right) \cdot \frac{\left(\vec{\frac{\delta r}{\delta s}} \times \vec{\frac{\delta r}{\delta t}}\right)}{\left\|\left(\vec{\frac{\delta r}{\delta s}} \times \vec{\frac{\delta r}{\delta t}}\right)\right\|} \cdot \left[\left(\left\|\vec{\frac{\delta r}{\delta s}} \times \vec{\frac{\delta r}{\delta t}}\right\|\right) \cdot dsdt\right] = \vec{F}\left(\vec{r(s,t)}\right) \cdot \left(\vec{\frac{\delta r}{\delta s}} \times \vec{\frac{\delta r}{\delta t}}\right) \cdot dsdt$$

$$\Phi = \int \int \vec{F}\left(\vec{r(s,t)}\right) \cdot \left(\vec{\frac{\delta r}{\delta s}} \times \vec{\frac{\delta r}{\delta t}}\right) ds\, dt$$

Where the double integral is taken over the parameter space.

Note $\left(\vec{\frac{\delta r}{\delta s}} \times \vec{\frac{\delta r}{\delta t}}\right)$ must be in the same direction as the orientation of the surface so we may have to reverse the order of the cross product in a given problem.

In our example we have $\overrightarrow{r(s,t)} = \cos(t)\cdot\vec{i} + \sin(t)\cdot\vec{j} + s\cdot\vec{k}$

$0 \le t \le 2\pi$ and $0 \le s \le 2$ is oriented with outward normals

Suppose $\vec{F} = x\cdot\vec{i} + y\cdot\vec{j} + z\cdot\vec{k}$

$$\overrightarrow{\frac{\delta r}{\delta s}} \times \overrightarrow{\frac{\delta r}{\delta t}} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -\sin(t) & \cos(t) & 0 \end{pmatrix} = (-\cos(t))\cdot\vec{i} - \sin(t)\cdot\vec{j}$$

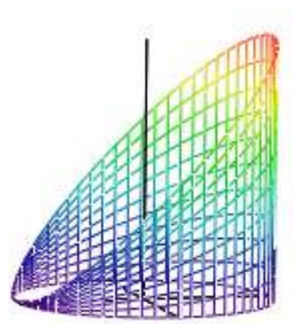
Since this would be an inward normal we use $\cos(t)\cdot\vec{i} + \sin(t)\cdot\vec{j}$

$$\vec{F} = x\cdot\vec{i} + y\cdot\vec{j} + z\cdot\vec{k} = \cos(t)\cdot\vec{i} + \sin(t)\cdot\vec{j} + s\cdot\vec{k}$$

$$\Phi = \int\int \vec{F}\left(\overrightarrow{r(s,t)}\right)\cdot\left(\overrightarrow{\frac{\delta r}{\delta s}} \times \overrightarrow{\frac{\delta r}{\delta t}}\right) ds\,dt = \int_0^{2\cdot\pi}\int_0^2 1\,dt\,ds = 4\cdot\pi$$

Example 2 Suppose we have as or surface the lateral part of the cylinder $x^2 + y^2 = 1$

cut by the plane z = y +1. oriented with outward normals



Suppose $\vec{F} = 2\cdot\vec{i} + 3\cdot\vec{j} + 2\cdot\vec{k}$

We parameterize the surface with

$x = \cos(t)$ $y = \sin(t)$ $z = s\cdot(\sin(t) + 1)$ $0 \le t \le 2\pi$ $0 \le s \le 1$

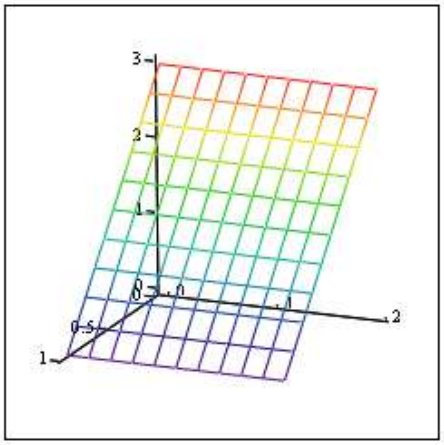
$$\vec{r} = \cos(t)\cdot\vec{i} + \sin(t)\cdot\vec{j} + s\cdot(\sin(t) + 1)\cdot\vec{k}$$

$$\overrightarrow{\frac{\delta r}{\delta t}} \times \overrightarrow{\frac{\delta r}{\delta s}} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(t) & \cos(t) & s\cdot(\cos(t)) \\ 0 & 0 & \sin(t) + 1 \end{bmatrix} = (\cos(t)\cdot\sin(t) + \cos(t))\cdot\vec{i} + \left(\sin(t)^2 + \sin(t)\right)\cdot\vec{j}$$

$$\Phi = \int\int \vec{F}\left(\overrightarrow{r(s,t)}\right)\cdot\left(\overrightarrow{\frac{\delta r}{\delta s}} \times \overrightarrow{\frac{\delta r}{\delta t}}\right) ds\, dt = \int_0^{2\cdot\pi}\int_0^1 2\cos(t)\sin(t) + 2\cdot\cos(t) + 3\cdot\sin(t)^2 + 3\cdot\sin(t)\, ds\, dt = 3\cdot\pi$$

Example 3

Let $\vec{F} = z\cdot\vec{i} + x\cdot\vec{j} + y\cdot\vec{k}$ and let S be the rectangle (1,0,0), (1,2,0), and (0,0,3),and (0,2,3) oriented with an upward normal



(x2, y2, z2)

To get our 2 vectors we consider the vector from (1,0,0) to (1,2,0) which is $2\cdot\vec{j}$

and the vector from (1,0,0) to (0,0,3) which is $-\vec{i} + 3\cdot\vec{k}$

For our fixed point we choose (1,0,0) both s and t vary from 0 to 1

$x = 1 - t$ $\qquad$ $y = 2\cdot s$ $\qquad$ $z = 3t$

$$\overrightarrow{\frac{\delta r}{\delta s}} \times \overrightarrow{\frac{\delta r}{\delta t}} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} = 6\cdot\vec{i} + 2\cdot\vec{k}$$

$$\vec{F} = z\cdot\vec{i} + x\cdot\vec{j} + y\cdot\vec{k} = 3\cdot t\cdot\vec{i} + (1 - t)\cdot\vec{j} + 2\cdot s\cdot\vec{k}$$

$$\vec{F}\left(\overrightarrow{r(s,t)}\right)\cdot\left(\overrightarrow{\frac{\delta r}{\delta s}} \times \overrightarrow{\frac{\delta r}{\delta t}}\right) = 18\cdot t + 4\cdot s$$

$$\Phi = \int\int \vec{F}\left(\overrightarrow{r(s,t)}\right)\cdot\left(\overrightarrow{\frac{\delta r}{\delta s}} \times \overrightarrow{\frac{\delta r}{\delta t}}\right) ds\, dt = \int_0^1\int_0^1 18\cdot t + 4\cdot s\, ds\, dt = 11$$

Surface Area

Recall $\int 1\, dS$ = Surface Area . For Parameterized Surfaces This takes the Form

$$S = \int\int \left(\overrightarrow{\frac{\delta r}{\delta s}} \times \overrightarrow{\frac{\delta r}{\delta t}}\right) ds\, dt$$

Find the Area of the rectangle in example 3

$$\int\int\left(\frac{\delta\vec{r}}{\delta s}\times\frac{\delta\vec{r}}{\delta t}\right)ds\,dt=\int_0^1\int_0^1 2\cdot\sqrt{10}\,ds\,dt=2\cdot\sqrt{10}$$

You can verify this with the appropriate cross product