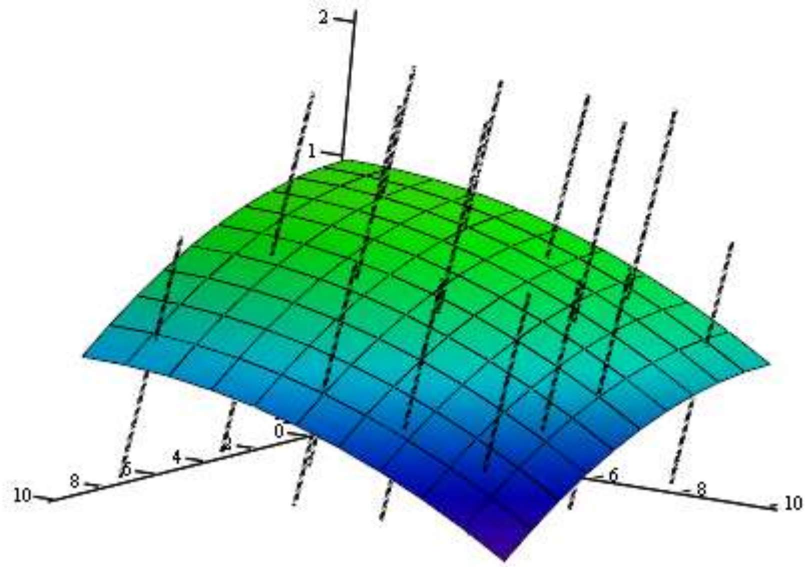
Flux Integrals

Suppose we have a surface σ given by z = f(x,y)

and a vector field $\vec{v} = f_1(x,y,z)\cdot\vec{i} + f_2(x,y,z)\cdot\vec{j} + f_3(x,y,z)\cdot\vec{k}$

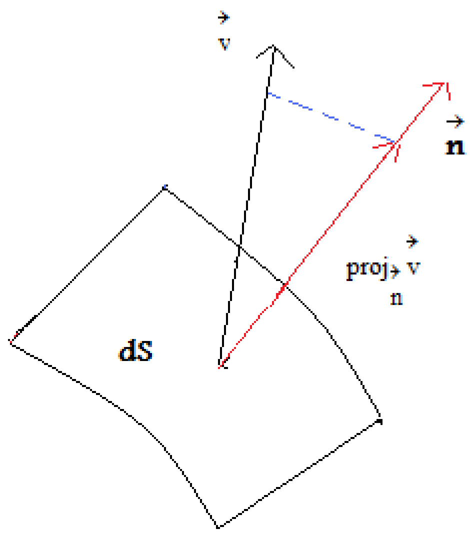


We develop the flux integral Φ .

We partition the surface into a large number of rectangular patches with area ds. On each of these patches $\vec{v}$ is approximately constant. We then project $\vec{v}$ onto $\vec{n}$ the unit normal.

We then have the situation where we have a flow perpendicular to a flat surface therefore

$$d\Phi = \left\|\mathrm{proj}_{\vec{n}}\vec{v}\right\|\cdot ds = \vec{v}.\vec{n}\cdot ds$$



It follows then $\Phi = \iint_{\sigma} \vec{v} . \vec{n} \cdot ds$

But this is simply a surface integral which we already know how to integrate i.e.

$$\iint_{\sigma} \vec{v} . \vec{n} \cdot ds = \iint_{\mathbf{R}} \vec{v} \cdot \vec{n} \cdot \sqrt{1 + \left(\frac{\delta f}{dx}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2}\, dA$$

Where R is the region in the x--y plane over which the surface σ lies.

Recall from our work on tangent planes the normal (upward) to the surface is $\vec{N} = \frac{-\delta f}{\delta x} \cdot \vec{i} - \frac{\delta f}{\delta y} \cdot \vec{j} + 1.$

But $\vec{n} = \frac{\vec{N}}{\left|\left|\vec{N}\right|\right|} = \frac{\frac{-\delta f}{\delta x} \cdot \vec{i} - \frac{\delta f}{\delta y} \cdot \vec{j} + 1}{\sqrt{1 + \left(\frac{\delta f}{dx}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2}}$

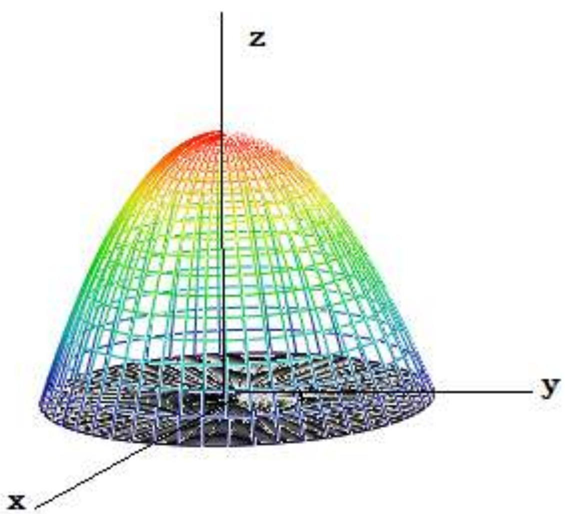
Therefore the flux integral is given by :

$$\Phi = \int\int_{\sigma} \vec{v}.\vec{n} \cdot ds = \int\int_{R_{x,y}} \vec{v} \cdot \vec{N}\, dA$$

Where $\vec{N}$ is the upward normal to the surface given by z = f(x.y) where $\vec{N} = \frac{-\delta f}{\delta x} \cdot \vec{i} - \frac{\delta f}{\delta y} \cdot \vec{j} + 1.$ And $R_{x,y}$ is the region of integration in the x-y plane . Of course we can also use a downward normal and we can adjust $\vec{N}$ if the surface is given by x = f(y,z) or y = f(x,z) and we replace $R_{x,y}$ by $R_{z,y}$ or $R_{x,z}$ accordingly.

Example 1

Let $\vec{v} = x \cdot \vec{i} + y \cdot \vec{j} + z \cdot \vec{k}$ and σ be the portion of the parabaloid $z = 1 - x^2 - y^2$ above the x-y plane oriented with upward normals.



$$\vec{N} = 2 \cdot x \cdot \vec{i} + 2 \cdot y \cdot \vec{j} + \vec{k}$$

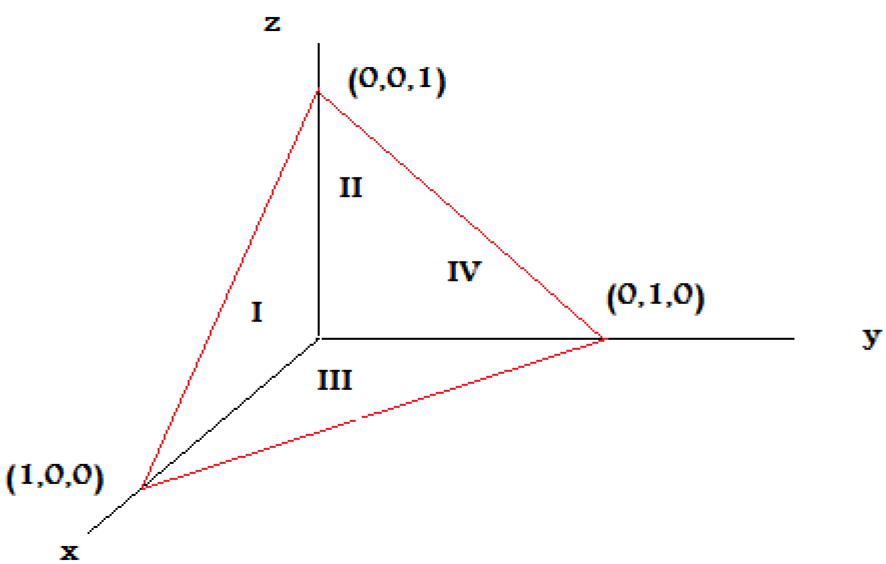
$$\vec{v} \cdot \vec{N} = 2 \cdot x^2 + 2 \cdot y^2 + z = 2 \cdot x^2 + 2 \cdot y^2 + 1 - x^2 - y^2 = x^2 + y^2 + 1$$

$$\Phi = \int\int_{\sigma} \vec{v} . \vec{n} \cdot ds = \int\int_{\mathbf{R}_{\mathbf{x,y}}} x^2 + y^2 + 1 \, dA = \int_0^{2\cdot\pi} \int_0^1 \left(r^2 + 1\right) \cdot r \, dr \, d\theta \rightarrow \frac{3 \cdot \pi}{2}$$

Example 2

Let $\vec{v} = x \cdot \vec{i} + y \cdot \vec{j} + z \cdot \vec{k}$ and σ be the tetrahedron x+y+z =1 in the first octant oriented with outward normals..

Here we have 4 faces to consider: Let I be the left face,II the back, III the bottom and IV the slant face.



We'll consider the 4 faces separately

Let's start with face I.

Here the region of integration is the triangular region in the x-z plane. The equation of this surface is y = 0, and an the outward normal is a left normal $\vec{N} = \frac{\delta y}{\delta x} \cdot \vec{i} - \vec{j} + \frac{\delta y}{\delta z} \cdot \vec{k} = -\vec{j}$

$\vec{v} \cdot \vec{N} = -y = 0$ therefore

$$\Phi = \iint_{\sigma} \vec{v} . \vec{n} \cdot ds = 0$$

For Face II the region of integration is the triangular region in the y-z plane . The equation of this surface is x = 0, and an the outward normal is a back normal $\vec{N} = -1 \cdot \vec{i} + \frac{\delta x}{\delta x} \cdot \vec{j} + \frac{\delta x}{\delta z} \cdot \vec{k} = -\vec{i}$

$\vec{v} \cdot \vec{N} = -y = 0$ therefore

$$\Phi = \iint_{\sigma} \vec{v} . \vec{n} \cdot ds = 0$$

I'll leave it for you to verify that the flux through the bottom is also 0 analogous to the first 2.

This leaves face IV, z= 1 - x - y. an the outward normal is an upward normal

$$\vec{N} = -\frac{\delta z}{\delta x} \cdot \vec{i} - \frac{\delta z}{\delta y} \cdot \vec{j} + \vec{k} = \vec{i} + \vec{j} + \vec{k}$$

$$\vec{v} \cdot \vec{N} = x + y + z = x + y + 1 - x - y = 1$$

$$\Phi = \iint_{\sigma} \vec{v} . \vec{n} \cdot ds = \int_0^1 \int_0^{1-x} 1 \, dy \, dx = \frac{1}{2}$$

Note had we used the divergence theorem :

$$\operatorname{div} \vec{v} = 3$$

$$\iint_{\sigma} \vec{v} \cdot \vec{n} \cdot ds = \iiint_{\mathbf{G}} \mathbf{div}\, \vec{v} \cdot \mathbf{dv}$$

$$\iint_{\sigma} \vec{v} \cdot \vec{n} \cdot ds = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 3 \, dz \, dy \, dx \rightarrow \frac{1}{2}$$