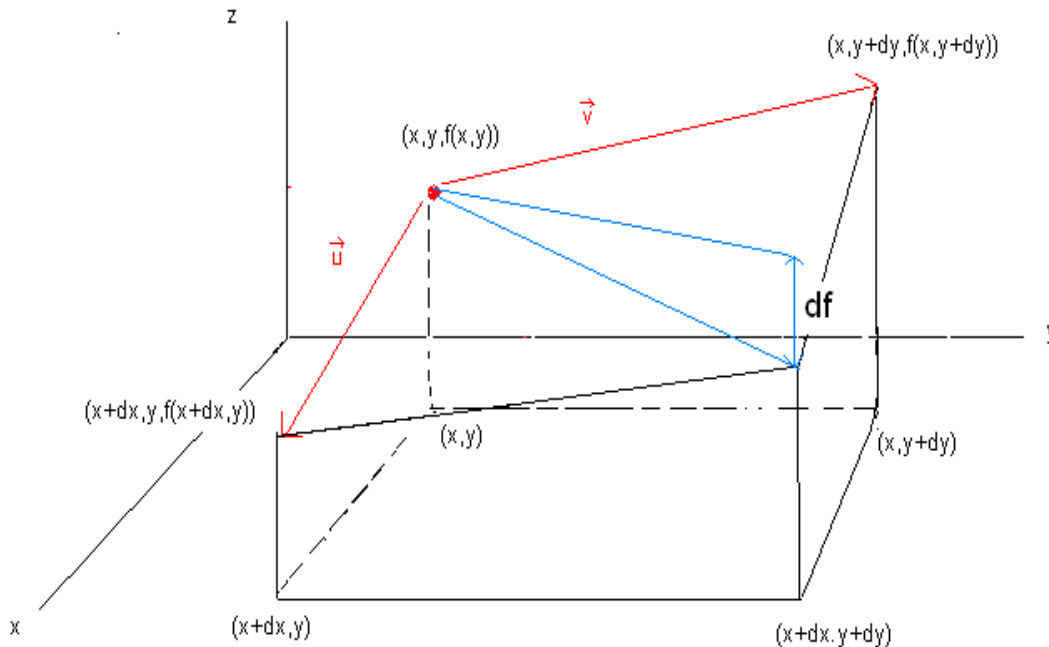


The Differential for Functions of 2 Variables

Recall that for functions of one variable we defined the differential as the change in f as we moved along the tangent line letting x change by a small amount dx .

We obtained $df = f'(x)dx$.

For functions of 2 variables we define the differential as the change in $f(x,y)$ as we move along the tangent plane as x and y change from (x,y) to $(x+dx,y+dy)$ where dx and dy represent small changes in x and y respectively.



Recall the equation of the tangent plane : $z = \frac{\delta f}{\delta x} \cdot (x - x_0) + \frac{\delta f}{\delta y} \cdot (y - y_0) + f(x_0, y_0)$

$$df = \left[\frac{\delta f}{\delta x} \cdot (x + dx - x) + \frac{\delta f}{\delta y} \cdot (y + dy - y) + f(x, y) \right] - f(x, y)$$

We then define the differential for functions of 2 variables as $df = \frac{\delta f}{\delta x} \cdot dx + \frac{\delta f}{\delta y} \cdot dy$ where δ represents the partial differentiation operator. Note the similarity to the definition for functions of one variable.

Example 1

Suppose $f(x,y) = x^2 \cdot \sin(y)$. Suppose we move from $(1,2)$ to the pt $(.95, 2.01)$ Use the differential to estimate the change in f .

$$\frac{\delta f}{\delta x} = 2 \cdot x \cdot \sin(y) \quad \text{and} \quad \frac{\delta f}{\delta y} = x^2 \cdot \cos(y) \quad dx = -.05 \quad \text{and} \quad dy = .01$$

$$df = 2x\sin(y)dx + x^2 \cos(y)dy$$

Therefore $df=2\cdot\sin(2)\cdot(-.05) + \cos(2)\cdot(.01) = -0.095$

The actual change is $(.95)^2 \sin(2.01) - 1^2\cdot\sin(2) = -0.092$

Of Course this example does not reflect what the differential is used for but should give us some confidence in its effectiveness.

In application the differential is used in 2 types of applications.

1. Error Analysis
2. Estimating the response of the output to small changes in the inputs.

Error Analysis

Just as for functions of one variable df represents an estimation in the calculated error in f based on the errors in measurements, dx and dy , of x and y .

Example 2

For example suppose we want to calculate the volume of a cylinder. We measure the radius to be 9 cm with a maximum possible error of $\pm .2$ cm and the height to be 7cm with a max possible error of $\pm .5$ cm.

Then $dr = 0.2$ and $dh = 0.5$

What is the maximum possible error in the calculation of f ?

Here we use in General $df = \left| \frac{\delta f}{\delta x} \cdot dx \right| + \left| \frac{\delta f}{\delta y} \cdot dy \right|$

The formula for the volume of a cylinder is $V = \pi \cdot r^2 \cdot h$

The differential takes the form $dV = \left| \frac{\delta V}{\delta r} \cdot dr \right| + \left| \frac{\delta V}{\delta h} \cdot dh \right|$

$$dV = \left| 2 \cdot \pi \cdot r \cdot h \cdot dr \right| + \left| \pi \cdot r^2 \cdot dh \right| = \left| 2 \cdot \pi \cdot 9 \cdot 7 \cdot (.2) \right| + \left| \pi \cdot 9^2 \cdot (.5) \right| = 206.403 \text{ cm}^3.$$

So the question is this a large error?

This brings us again to the notion of relative and % errors.

The relative error is $\frac{df}{f}$ and the % error is the relative error x 100 = $\frac{df}{f} \times 100$.

In our example $\frac{df}{f} = \frac{dV}{V} = \frac{206.403}{\pi \cdot 9^2 \cdot 7} = .116$ or an error of 11.6%.

You may have wondered to get the max volume we could calculate the V with max values of r and h and to calculate the min volume we could have calculated V using the min values of r and h. The answer is we could have. However suppose we had a function

like $f(x, y) = \frac{\cos(x) \cdot \sin(y) \cdot x}{x \cdot \ln(y) \cdot \sec(y)}$ Now it is not clear at all if the max values of x and y give the

max value of f or if the max value of x and the min value of y gives the max value of f or any of the possible permutations, in fact none of these may give the max value.

But at a point of differentiability we always have local linearity and the differential always works!

Example 3 Estimating the response of the output to small changes in the inputs.

This works just like example 1 but this time we'll do an application.

The ideal gas law states that $P = \frac{kT}{V}$ where P is the pressure, T is the temperature, V is the volume, and k is a constant.

Suppose the Pressure is 1.17 atmospheres when the volume is 1.4 liters and the temperature is 300degK.

- Find an expression for dP for changes dT and dV.
- Find an expression for the relative change in Pressure
- Suppose we decrease T by 5K and decrease the Volume by .05 l. What is the % change in Pressure?
- If the temperature decrease by 10K By How much should the volume decrease to maintain constant pressure

a. $df = \frac{\delta f}{\delta x} \cdot dx + \frac{\delta f}{\delta y} \cdot dy$ takes the form $dP = \left(\frac{\delta P}{\delta T} \cdot dT + \frac{\delta P}{\delta V} \cdot dV \right) = \frac{k}{V} \cdot dT - k \cdot \frac{T \cdot dV}{V^2}$

$$b. \frac{dP}{P} = \frac{\frac{k}{V} \cdot dT - k \cdot \frac{T \cdot dV}{V^2}}{\frac{kT}{V}} = \frac{dT}{T} - \frac{dV}{V} \quad \text{which are precisely the relative changes in } T \text{ and } V.$$

This reflects that an increase in temperature increases the pressure while an increase in volume decreases the pressure.

$$c. \frac{dP}{P} = \left(\frac{5}{300} - \frac{0.05}{1.4} \right) = -0.019 \text{ the pressure decreases by } 1.9\%.$$

d. We want $dP = 0$ therefore $\frac{-10}{300} - \frac{dV}{1.4} = 0$ Solving for dV we get $dV = -.047$ l So if we decrease T by 10K and the volume decreases by .047 liters the pressure will remain the same .

We are also now in a position to develop one version of the chain rule for functions of 2 variables:

If $f = f(x,y)$ and $x = x(t)$ and $y = y(t)$ Then it follows

$$\frac{df}{dt} = \frac{\delta f}{\delta x} \cdot \frac{dx}{dt} + \frac{\delta f}{\delta y} \cdot \frac{dy}{dt}$$

This gives us the rate at which $f(x,y)$ is changing at a point as we move along the curve parameterized by

$x = x(t)$ and $y = y(t)$ in the domain . See the lectures on Parametric Eqns in 3-Sapce and Directional Derivatives

One last note : For functions of 3 variables we then define the differential as:

$$df = \frac{\delta f}{\delta x} \cdot dx + \frac{\delta f}{\delta y} \cdot dy + \frac{\delta f}{\delta z} \cdot dz$$

And the corresponding chain rule.

$$\frac{df}{dt} = \frac{\delta f}{\delta x} \cdot \frac{dx}{dt} + \frac{\delta f}{\delta y} \cdot \frac{dy}{dt} + \frac{\delta f}{\delta z} \cdot \frac{dz}{dt}$$