The Differential

Almost from the time when we first learned that the derivative of a function f(x) at a point x = a is the slope of the tangent line at the point (a, f(a) we calculated the equation of the tangent line at that point.

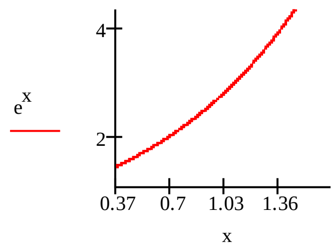
Specifically y = f ' (a) (x-a) +f(a). From now on we will use the notation l(x) for the tangent line i.e.
l(x) = f ' (a) (x-a) +f(a).

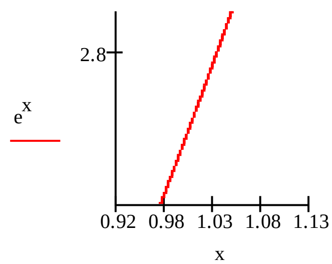
You may have and should have wondered why we were so interested in the tangent line itself. After all we haven't done anything with the tangent line once we calculated it, it has been the slope of the tangent line that we have used.

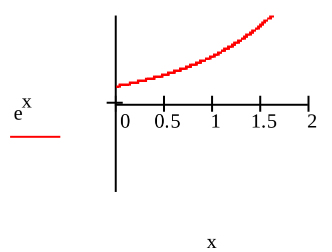
Tonight we see why we have been doing all that work.

First recall the notion of local linearity that we saw in our first lab. We say a function is locally linear at a point if on zooming in near a point the function tended to become linear. We could almost use this as a definition of differentiability i.e. a function is differential at a point if it is locally linear at that point.

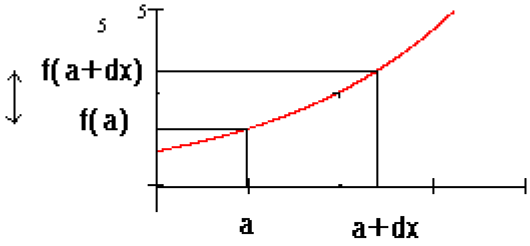
See the figures below where we zoom in on f(x) = e^x at x = 1:





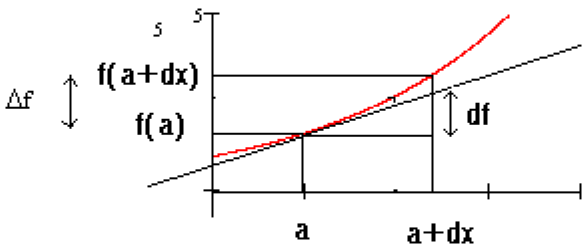


Suppose we change the input by a very small amount from x = a to x = a +dx. Then the change in output Δf = f(a+dx) - f(a) is the change in f as we move along the curve y = f(x). We can think of Δf as the response of the output to a small change in the input.



Suppose instead of moving along the curve we use the tangent line l(x) = f ' (a) (x-a) +f(a).

We define the differential of f(x),denoted df, as the change in f as we move along the tangent line allowing x to change by an amount dx from x = a to x = a + dx.



How exactly do we compute the differential ?
l(a+dx) - l (a) = f ' (a) (a+dx - a) +f(a) - f(a) = f ' (a) dx

i.e. The differential of a function at a point x is df = f '(x) dx.
Example Let f(x) = sin(x) Then df = cos(x) dx

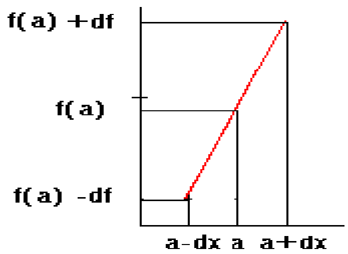
Let f(x) = ln(x) Then df = $\frac{1}{x}$dx.

How is the differential used in applications?

Suppose f = f(x) but suppose x is a measured quantity. There are no perfect measurements therefore there will be an uncertainty in the measurement of x. Therefore there will be an uncertainty in the calculated value of f(x).
The question is then How will an uncertainty in the measurement of x propagate through in the calculation of f(x)?

Denote the uncertainty in the measurement of x by dx. Note in general the uncertainty in the measurement of a quantity will be much less than the measurement. We wouldn't use a yardstick to measure the diameter of a molecule.

What is the maximum uncertainty in the calculation of f(x) ? Here we use the local linearity and the differential.



We see that the maximum uncertainty in the calculation of f(x)
is df the differential of f(x), (again df is the change in f as we move along the tangent line)

Summary

1. The differential of f(x) is the change in f(x) as we move along the tangent line allowing x to change by an amount dx

2. df = f ' (x) dx

3. df is the uncertainty in the calculation of f(x) due to an uncertainty, dx, in the measurement of x.
Example Suppose we want to calculate the Area of a circle A by measuring the radius R.
Suppose we measure the radius to be 7 cm with an uncertainty of $\pm$ 0.1cm. What is the uncertainty in the Calculation of the Area ?

$A(R) = \pi R^2$.
$dA = A'(R)\, dR = 2\pi R dR = 2\pi(7)(.1) = 1.4\,\pi$. Therefore our calculation of the area could be off by as much as
$1.4\,\pi$ cm^2 which is approximately 4.4 cm^2.

Is this a lot of error ? If we were off by 50 lbs in estimating the weight of an elephant we would probably say we had a good estimation. However if we were off by 50 lbs in estimating the weight of a chicken we would not have a very good estimation.

With this in mind we make the following definitions:

1. Relative Error = $\frac{df}{f}$.

2. Percentage Error $\frac{df}{f}$ x100

In our example dA = 4.4cm^2 and $A(7) = \pi 7^2 = 154$ cm^2.

Therefore the relative error is $\frac{dA}{A} = \frac{4.4}{154}$ = 0.029.
The percentage error is 2.9%.
Example Suppose we want to Calculate The volume of a sphere with an error of no more than 5%. What is the maximum possible percentage error in the measurement of the radius ? If the radius is measured to be 5.5 cm with a max possible error of 0.2cm will the volume be within 5%?

$V(r) = \frac{4}{3} \cdot \pi \cdot r^3$ Then $dV = 4 \cdot \pi \cdot r^2\, dr$.

Then $\frac{dV}{V} = \frac{4\cdot\pi\cdot r^2\cdot dr}{\frac{4}{3}\cdot\pi\cdot r^3} = 3\cdot\frac{dr}{r}$. It follows $\frac{dr}{r} = \frac{1}{3}\cdot\frac{dV}{V} = \frac{1}{3}\cdot .05 = .017$ therefore the max possible % error in the measurement of the radius can be no more than 1.7%

In our example $\frac{.2}{5.5} = 0.036$ therefore our measurement would not be accurate enough.
Example Using the differential to estimate the change in a quantity.

The ideal gas law states that $P = \frac{kT}{V}$ where P is the pressure, T is the temperature, V is the volume, and k is a constant.

Suppose the Pressure is 1.17 atmospheres when the volume is 1.4 liters and the temperature is 300degK.

a. If the volume is held constant and the temperature increases by 5 deg K what is the % change in Pressure?

b. If the temp is held constant and the volume increases by .2 liters what is the change in Pressure ?

a. dP = P ' (T)dT = $\frac{k}{V}dT$.

$$\frac{dP}{P} = \frac{\frac{kdT}{V}}{\frac{kT}{V}} = \frac{dT}{T} = \frac{5}{300} = .017$$. Therefore the pressure increase by 1.7%

b. dP = P '(V) dV = $\frac{-k\cdot T}{V^2}dV$.

$$\frac{dP}{P} = \frac{\frac{-k\cdot T\cdot dV}{V^2}}{\left(\frac{k\cdot T}{V}\right)} = \frac{-dV}{V} = \frac{.2}{1.4} = -.143$$. Therefore the pressure decreases by 14.3%

Supplemental Homework - The Differential

1. The side of a square is measured to be 10ft with an uncertainty of $\pm$ 0.1 ft in the measurement of the side.

   a. Use differentials to estimate the error in the calculated area.

   b. Estimate the percentage errors in the measurement of the side and the calculation of the area.

Ans: a. $\pm$ 2ft^2 b. side 1% area 2%.

2. The side of a cube is measured to be 25 cm with a possible error of $\pm$ 1cm.

   a. Use differentials to estimate the error in the calculation of volume.

b. Estimate the percentage error in the side and the volume.

Ans: a. 1875cm^3 b. side 4% volume 12 %

3. The hypotenuse of a right triangle is known to be 10 in exactly. One of the acute angles is measured to be 30 degrees with a possible error of $\pm$ 1 degree. (Make sure to convert to radians)

a. Use differentials to estimate the errors in the calculation of the sides opposite and adjacent to the measured angle.

b. Estimate the percentage errors in the sides.

Ans : a. opp .151" adj .087" b. opp 3% adj 1%

4. The electrical resistance R of a wire is given by $R=\frac{k}{r^2}$ where k is a constant and r is the radius of the wire.
Assuming the radius has a possible error of $\pm$ 5%, use differentials to estimate the percentage error in the calculation of resistance.

Ans :10%

5. The side of a cube is measured with a possible percentage error of $\pm$ 2%. Use differentials to estimate the error in the calculation of the volume.

Ans : 6%

6. The area of a circle is to be computed from the measurement of its radius. Estimate the maximum permissible percentage error in the measurement of the side if the percentage error in the calculation of the area is to be less than $\pm$ 1 %.

Ans: 0.5%