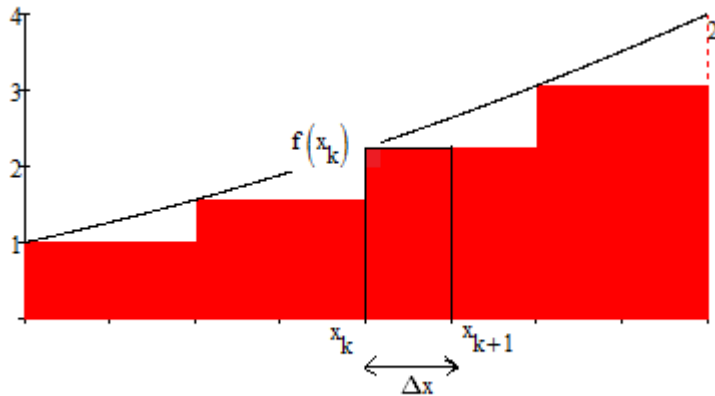


The Iterated Double Integral - A Riemann Sum Approach

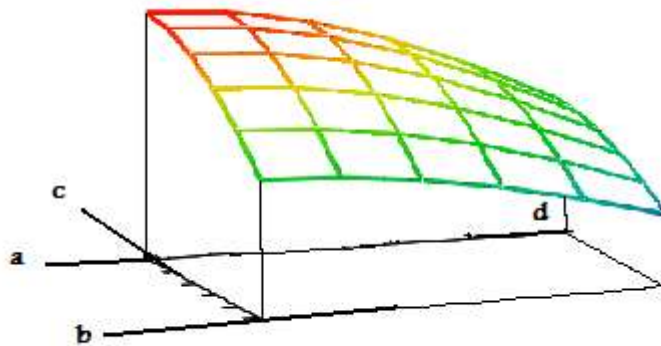
As with so many things in multivariable Calculus it is beneficial to return to single variable Calculus.

Let $f(x)$ be a function of one variable. To calculate the Area we partition the interval $[a,b]$ into a large number of subintervals of width Δx and form the Riemann Sum $\sum_{k=1}^n (f(x_k) \cdot \Delta x)$ which is really nothing more than a sum of rectangles. We then define the Area as the limit of this sum as the number of rectangles goes to ∞ i.e.

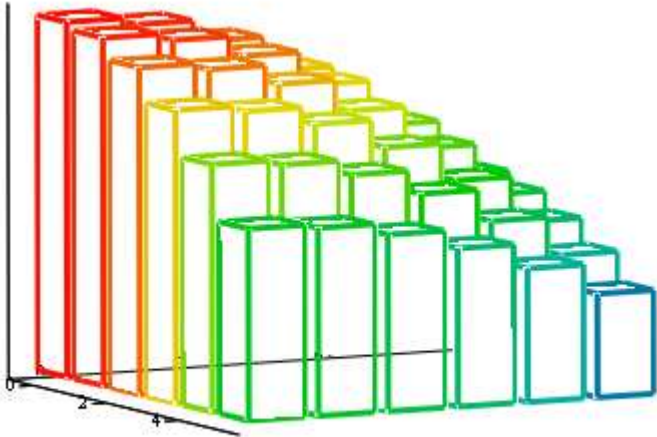
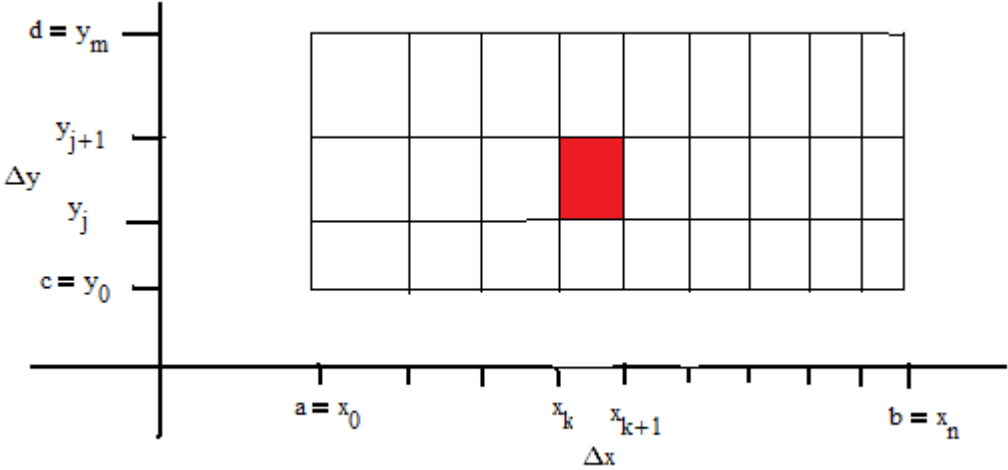
$$A = \lim_{n \rightarrow \infty} \sum_{k=1}^n (f(x_k) \cdot \Delta x) \quad . \text{ See animation 6.}$$



So Let $f(x,y)$ be a function of 2 variables defined over a rectangular domain $[a,b] \times [c,d]$.



We then partition the domain into a large number of rectangles of area $\Delta A = \Delta x \Delta y$. On each rectangle use the value of $f(x_k, y_j)$ to approximate the value of $f(x, y)$ on this rectangle.



The volume of the solid over each small rectangle is $\Delta V = f(x_k, y_j) \Delta A = f(x_k, y_j) \Delta x \Delta y$

Now we simply sum these volumes and then let the number of these rectangles go to ∞ .

[See Animation 7.](#)

The question is how do we sum over these rectangles ?

We start by fixing k and summing over j : $\sum_{j=1}^m (f(x_k, y_j) \cdot \Delta y)$ [See animation 8.](#)

We then sum over k $\sum_{k=1}^n \left[\sum_{j=1}^m (f(x_k, y_j) \cdot \Delta y) \right] \cdot \Delta x = \sum_{k=1}^n \sum_{j=1}^m (f(x_k, y_j) \cdot \Delta y \cdot \Delta x)$

What does this double sum look like ? [See Animation 9.](#)

Note the first sum approximates the volume of a cross-section of width Δy then the second sum adds the volumes of these cross-sections. This is precisely what we saw in the lecture Double Integrals-A

Cross-sectional Area approach !.

We then define the Volume as the limit by first taking the limit as m goes to ∞ and then the limit as n goes to ∞ . [You may want to view Animation 7 again.](#)

$$V = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{k=1}^n \sum_{j=1}^m (f(x_k, y_j) \cdot \Delta y \cdot \Delta x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \lim_{m \rightarrow \infty} \left[\sum_{j=1}^m (f(x_k, y_j) \cdot \Delta y \cdot \Delta x) \right]$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \lim_{m \rightarrow \infty} \left[\sum_{j=1}^m (f(x_k, y_j) \cdot \Delta y \cdot \Delta x) \right] = \lim_{n \rightarrow \infty} \left[\left(\sum_{k=1}^n \int_c^d f(x_k, y) dy \right) \cdot \Delta x \right] = \int_a^b \int_c^d f(x, y) dy dx$$

Note we generally use the symbol:

$$\iint f(x, y) dA$$

to represent the double integral where in Rectangular coordinates $dA = dx dy$ and on a rectangular domain:

$$\iint f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx \quad \text{or} \quad \int_c^d \int_a^b f(x,y) dx dy$$

In the next lecture we'll explore what happens if we have a non-rectangular domain.

One final and very important note. We saw with functions of one variable the use of the definite integral

extends way beyond the calculation of area. For example it is used in the calculation of distance, displacement, arclength, work and so on.

Similarly the double iterated integral calculates more than just mere volume. For example suppose the domain is a thin plate and $f(x,y)$ represents the density at each point. Then the double integral calculates the mass of the plate.

To see this if $f(x,y)$ is the density it is the mass/area . This is an areal density as the thickness is negligible.

Then over each rectangle in the partition the density is nearly constant and equal to $f(x_k, y_j)$. Here we are assuming the continuity of $f(x,y)$ which is a reasonable assumption based on physical grounds.

So the infinitesimal mass of this one rectangle is $dM = f(x_k, y_j) dA$. The total mass is then the double integral over the entire domain.

$$M = \iint_{\mathbf{R}} dM = \iint_{\mathbf{R}} f(x,y) dA$$

