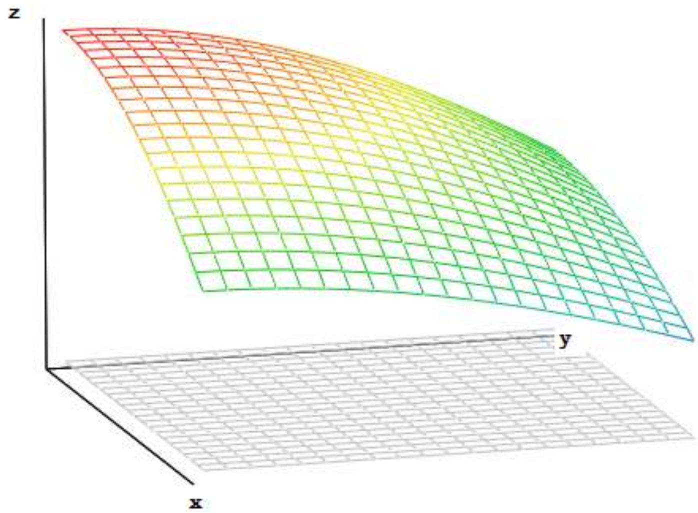
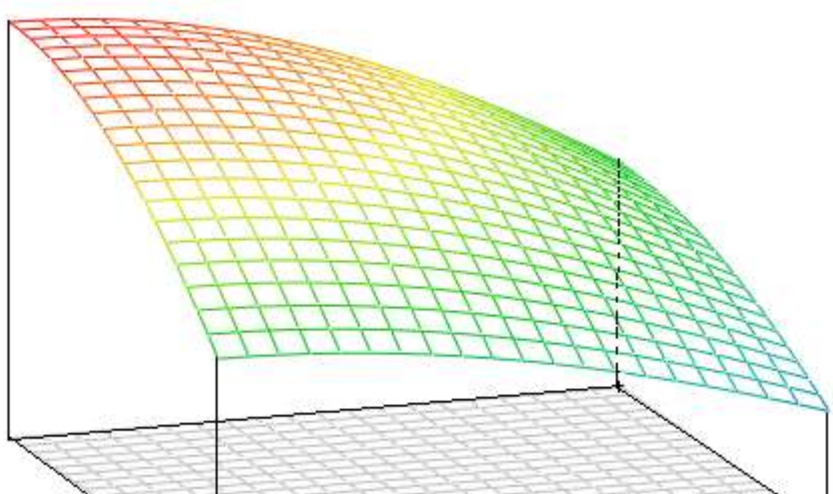
Double Integral - Cross- Sectional Area Notes

Suppose $z = f(x,y)$ is a surface over a rectangular domain R.



We can form a solid bounded above by the surface and below by the xy plane.

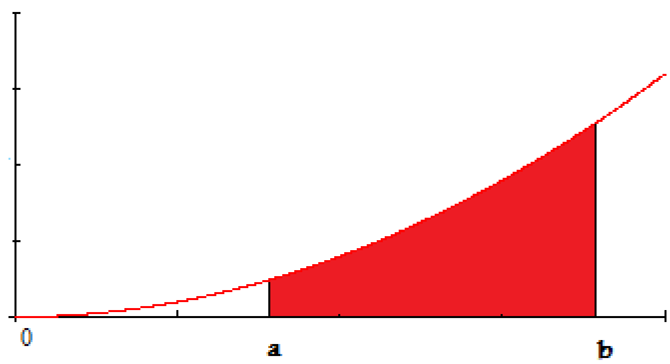


The question then is what is the volume?

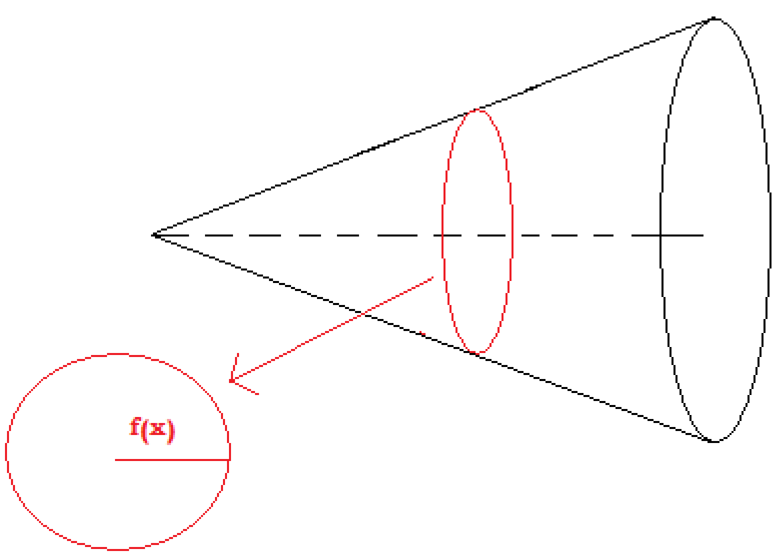
The answer lies in what you already know about volumes and areas from Calculus 1 and Calculus 2.

So let's review some basic ideas before answering the main question.

1.Of course you remember the area of the region between f(x) and the x-axis on the interval [a,b] is the definite integral $\int_a^b f(x)\,dx$.



2. In calculus 2 you learned that if the cross-sectional area of a solid is known then the volume is the definite integral of the cross-sectional area --- $\int_a^b A(x)\,dx$

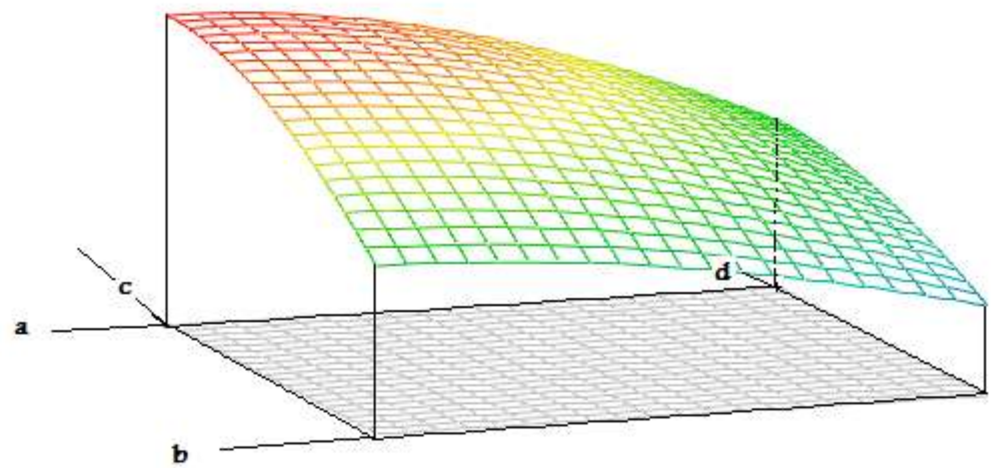


In particular for solids of revolution the cross-sectional areas are circles $A(x) = \pi f(x)^2$

And $V = \int_a^b \pi f(x)^2 dx$. But all we need to know is the cross-sectional area as a function of x.

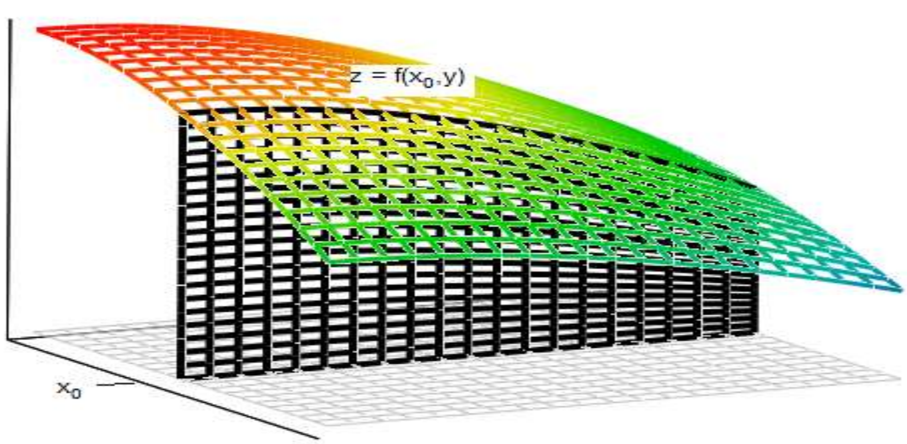
Recall the problems like the base of a solid is an ellipse and all cross-sections taken perpendicular to the base are equilateral triangles ?

So back to our original problem.



If we intersect our surface at any fixed value of x say x_0 we have as the curve of intersection

$z = f(x_0, y)$ which is a curve of a single variable. See Animation 1.



If we can figure out the cross-sectional area at each such x then the volume is simply

$$V = \int_a^b A(x)\,dx$$

Since the curve of intersection at x_0 $z = f(x_0,y)$ is a function of the single variable y it follows

$A(x_0) = \int_c^d f(x_0, y)\,dy$. Since this is true at every value of x we have $A(x) = \int_c^d f(x,y)\,dy$

The volume is then $V = \int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$

which is known as the iterated double integral. The first integration is done with respect to y

and the second integration is carried out with respect to x.

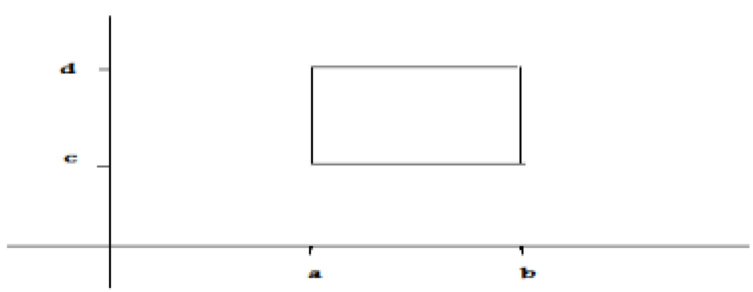
We can think of this as the inner integration calculates the areas of the cross-sections

and the outer integration adds up all these Areas.

See Animation 2.

Usually we only graph the domain to set-up the integrals. For Rectangular domains

We have



Example

Compute the volume of the solid bounded above by f(x,y) = cos(x)cos(y) over the square [0,1] x [0,1]

This is in fact the solid we have seen in the diagrams and animations.

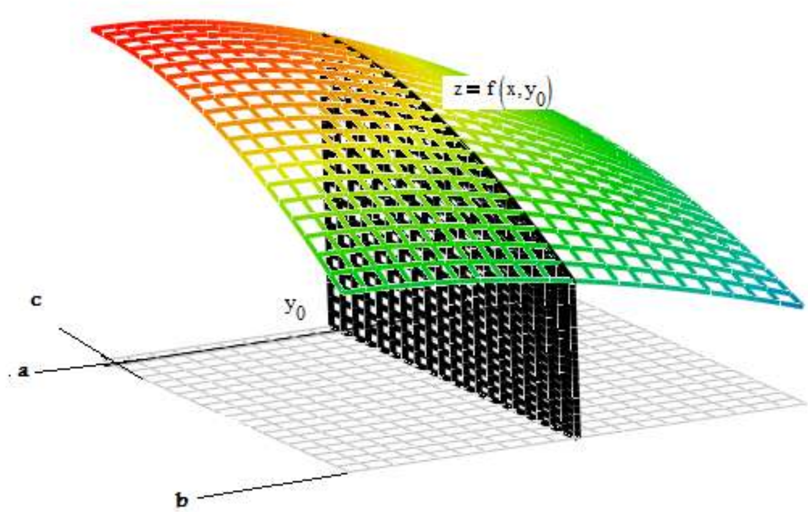
$$V = \int_0^1 \int_0^1 \cos(x) \cdot \cos(y)\, dy\, dx = \int_0^1 \cos(x)\left(\sin(y) \cdot \Big|_0^1 \right) dx = \int_0^1 \cos(x) \cdot \sin(1)\, dx = \sin(1) \sin(x) \cdot \Big|_0^1 = \sin(1)^2 = .7$$

The Order of Integration

We could have also set up the solution by fixing $y = y_0$ and having a curve of intersection $z = f(x,y_0)$

See Animations 3

See Animation 4.



This time we have $V = \int_a^b A(y)\, dy = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy = \int_c^d \int_a^b f(x,y)\, dx dy$