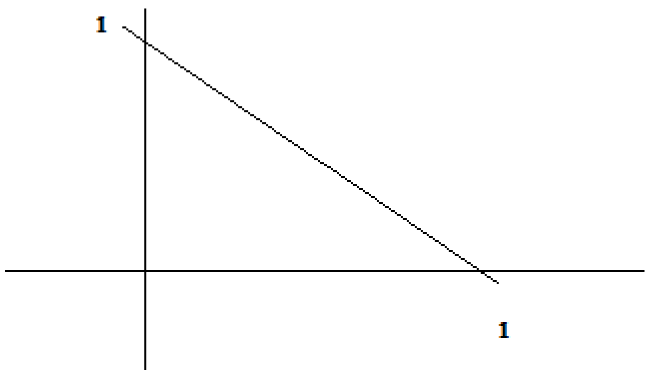
Using Transformations to evaluate a Double Integral

Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y}\cdot(y - 2\cdot x)^2\,dy\,dx$

You're region is



Make the transformations : Since u = x+y and v = y - 2x

To convert an integral you need 3 things

1. Convert the integrand to functions of u and v

Since u = x+y and v = y - 2x

We obtain

$$\int_{\blacksquare}^{\blacksquare} \int_{\blacksquare}^{\blacksquare} \sqrt{u}\cdot v^2\,d\blacksquare\,d\blacksquare$$

2. Using the Jacobian convert the areal element dxdy to Jdvdu where J is the value of the Jacobian

Solving for x and y in terms of u and v we have (at the very end I'll give you an easy matrix method of doing this if you have trouble)

$$x = \frac{u - v}{3} \qquad y = \frac{2 \cdot u + v}{3}$$

$$J = \begin{pmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta v} & \frac{\delta y}{\delta v} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{3}$$

Now we have

$$\frac{1}{3} \cdot \int \int \sqrt{u} \cdot v^2 \, dv \, du$$

3. Now the hard part usually is converting the region in the x-y plane to one in the u-v plane.

$$x = \frac{u - v}{3}$$

$$y = \frac{2 \cdot u + v}{3}$$

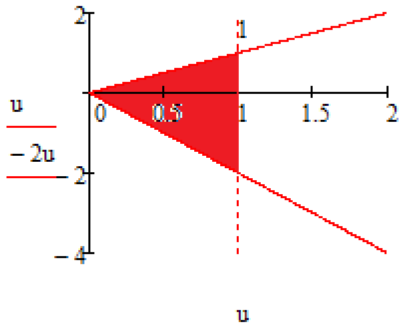
If x= 0 we have v = u

If y = 0 we have v = -2u

If y = 1-x we have $\frac{2 \cdot u + v}{3} = 1 - \frac{u - v}{3}$

Solving we get u = 1

So our region in the uv - plane is



So finally we obtain:

$$\frac{1}{3}\cdot\int_0^1\int_{-2\cdot u}^{u}\sqrt{u}\cdot v^2\,dv\,du = 0.222$$

Solving for x and y in terms of u and v

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}^{-1}\cdot\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{3}\cdot\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{3}\cdot\begin{pmatrix} u - v \\ 2\cdot u + v \end{pmatrix}$$