Lab 4 Using Contour Diagrams and Gradient Fields to Identify Extrema

As we learned in lecture the critical points of a function of two variables are obtained by the simultaneous solutions of the equations :

$$\partial f/ \partial x = 0$$

$$\partial f/\partial y = 0$$

Further we saw if (x_0,y_0) is a critical point then there are three possibilities. At (x_0,y_0) , f(x,y) could have a local maximum, a local minimum, or a saddle point.

The purpose of this lab is to use contour diagrams and gradient fields to classify the critical points.

Let's consider the example $f(x,y) = x^3 + y^3 - 3x -3y$.

Then $\partial f/ \partial x = 3x^2 - 3 = 0$ for $x = \pm 1$

$\partial f/\partial y = 3y^2 - 3 = 0$ for $y = \pm 1$

Therefore there are four critical points (1,1),(1,-1),(-1,1),(-1,-1).

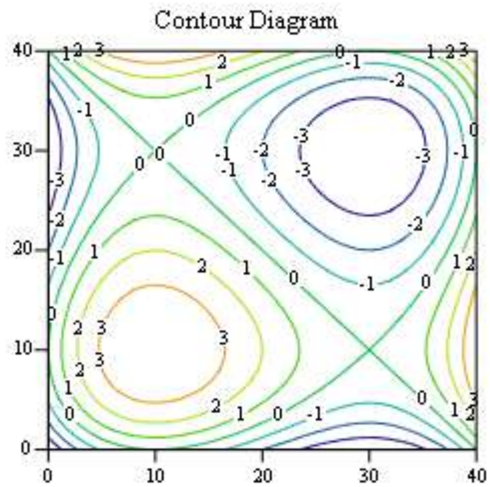
To classify the critical points consider the contour diagram on page 2 and the gradient field on page 3

$a := -2 \quad b := 2 \quad c := -2 \quad d := 2 \quad \Delta x := .1 \quad \Delta y := .1$

$i := 0 .. \frac{b - a}{\Delta x} \quad j := 0 .. \frac{d - c}{\Delta y} \quad x_i := a + i \cdot \Delta x \quad y_j := c + j \cdot \Delta y$

$f(x, y) := \left(x^3 + y^3 - 3 \cdot x - 3 \cdot y\right) \quad M_{i,j} := f\left(x_i, y_j\right)$

(Note the formatting is done exactly as in labs 1 and 2)



M

To identify the x and y values of the points recall i corresponds to x and j corresponds to y. Therefore to identify the coordinates x_i and y_j type x_i = and y_j =.

$x_{10} = -1$ $y_{30} = 1$ Saddle point : moving from (-1,1) vertically f increases but f decreases moving horizontally from (-1,1)

$x_{30} = 1$ $y_{30} = 1$ Minimum : f increases in all directions moving away from (1,1)

$x_{30} = 1$ $y_{10} = -1$ Saddle point : moving from (1,-1) vertically f decreases but f increases moving horizontally from (1,-1)

$x_{10} = -1$ $y_{10} = -1$ Maximum : f decreases in all directions moving from (-1,-1)

<u>Creating Gradient Fields</u>

The Formatting is a little different, The First 2 lines are the same but in the 3d line we define the partial derivatives of f(x,y).

We define two array elements: $X_{i,j}$ is $\partial f/\partial x$ and $Y_{i,j}$ is $\partial f/\partial y$. We Plot (X,Y) in the placeholder-

make sure to use the parentheses.

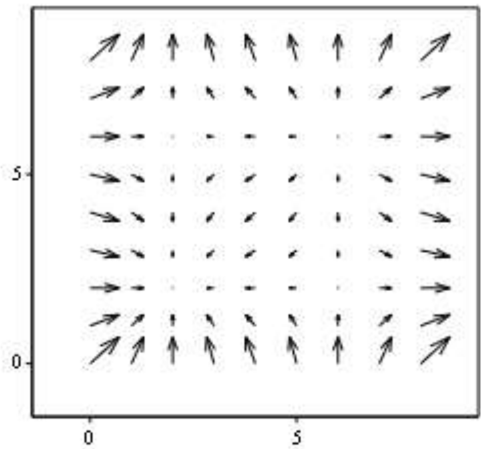
In the FORMAT window we choose Display As VECTOR FIELD PLOT

$a := -2 \quad b := 2 \quad c := -2 \quad d := 2 \quad \Delta x := .5 \quad \Delta y := .5$

$i := 0..\frac{b-a}{\Delta x} \quad j := 0..\frac{d-c}{\Delta y} \quad x_i := a + i\cdot\Delta x \quad y_j := c + j\cdot\Delta y$

$X_{i,j} := 3\cdot\left(x_i\right)^2 - 3 \quad Y_{i,j} := 3\cdot\left(y_j\right)^2 - 3$

Gradient Field



(X, Y)

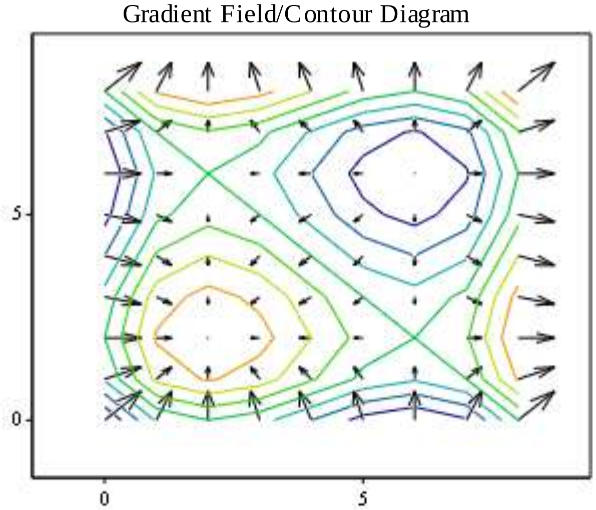
$a := -2 \quad b := 2 \quad c := -2 \quad d := 2 \quad \Delta x := .5 \quad \Delta y := .5$

$i := 0 .. \frac{b-a}{\Delta x} \quad j := 0 .. \frac{d-c}{\Delta y} \quad x_i := a + i \cdot \Delta x \quad y_j := c + j \cdot \Delta y$

$X_{i,j} := 3 \cdot \left(x_i\right)^2 - 3 \quad Y_{i,j} := 3 \cdot \left(y_j\right)^2 - 3 \quad f(x,y) := \left(x^3 + y^3 - 3 \cdot x - 3 \cdot y\right) \quad N_{i,j} := f\left(x_i, y_j\right)$

Note we simply added the formula for f and N to plot the Gradient Field and the contour Diagram.

Gradient Field/Contour Diagram



(X, Y), N

Recall that ∇f is the direction of greatest increase of f(x,y) and that it is perpendicular to level curves.

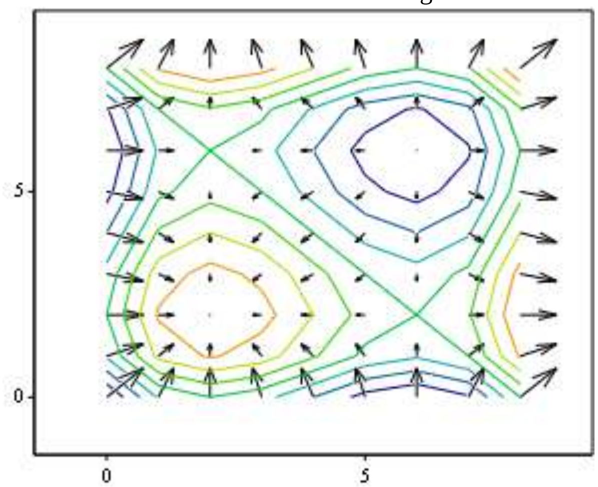
If the gradients of f point towards a critical point from all directions then f(x,y) has a maximum at (x,y);

If the gradients point away from a critical point in all directions then f(x,y) has a minimum at (x,y); and if

the gradients point toward a critical point in some directions and point away from the critical point in

perpendicular directions then f(x,y) has a saddle point at (x,y).

Gradient Field/Contour Diagram



(X, Y), N

$x_2 = -1$ $y_2 = -1$ all gradients point toward (-1,-1) therefore we have a maximum at(-1,-1)

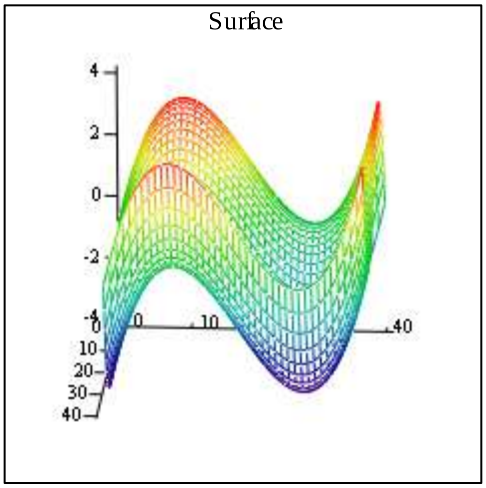
$x_6 = 1$ $y_6 = 1$ all gradients point away from (-1,-1) therefore we have a minimum at(-1,-1)

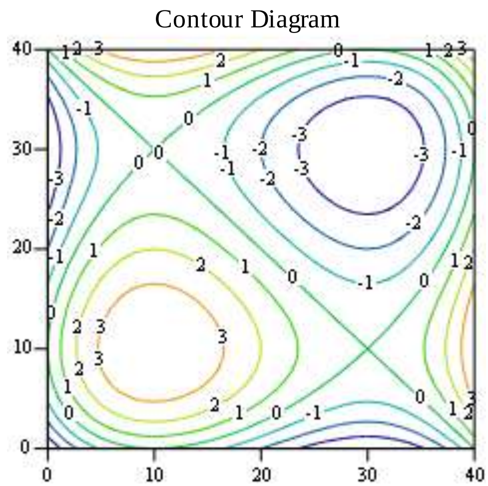
$x_2 = -1$ $y_6 = 1$ vertically the gradients point away from (-1,1) and horizontally they point toward (-1,1) therefore we have a Saddle Point

$x_6 = 1$ $y_2 = -1$ vertically the gradients point toward(-1,1) and horizontally they point away from (-1,1) therefore we have a Saddle Point

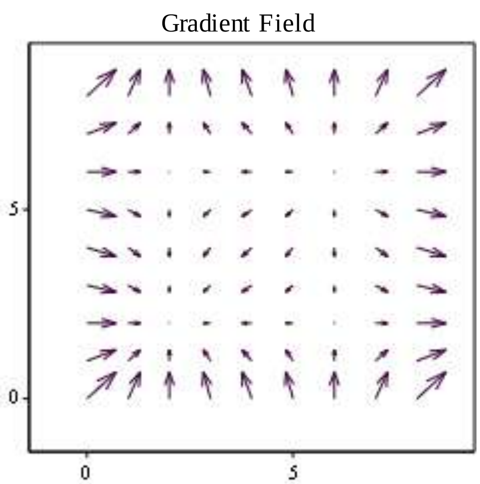
Finally let's compare the surface with its contour diagram, and the contour diagram and the gradient field

Recall for the surface x comes out of the page and for the contour the x axis is horizontal.





M M



(X, Y)

Exercise

Let f(x,y) = cos(x) sin(y) on [-2, 2] x [-2,2] (to graph use $\Delta x = \Delta y = .1$)

a. Identify the critical points. There are 4.

b. Classify the critical points as local maxima, minima, or saddle points by considering the contour plot.

c. Classify the critical points as local maxima, minima, or saddle points by considering the gradient field.

Use $\Delta X = \Delta Y = .5$ for the gradient field.

d. On a separate sheet of paper verify by hand that the critical points are $(0,\pi/2)$, $(0,-\pi/2)$,$(\pi/2,0)$,and $(-\pi/2,0)$

e. On a separate sheet of paper verify by hand using the second partials test the results you obtained graphically.