Define $g(x) = f\left(x + \frac{1}{n}\right) - f(x)$

$$g(0) = f\left(\frac{1}{n}\right) - f(0)$$

$$g\left(\frac{1}{n}\right) = f\left(\frac{2}{n}\right) - f\left(\frac{1}{n}\right)$$

$$g\left(\frac{2}{n}\right) = f\left(\frac{3}{n}\right) - f\left(\frac{2}{n}\right)$$

Continuing evaluating g(k/n)

$$g\left(1 - \frac{2}{n}\right) = f\left(1 - \frac{1}{n}\right) - f\left(1 - \frac{2}{n}\right)$$

$$g\left(1 - \frac{1}{n}\right) = f(1) - f\left(1 - \frac{1}{n}\right)$$

$$\sum_{k=0}^{n-1} g\left(\frac{k}{n}\right) = 0$$

If $g\left(\frac{k}{n}\right)$ = 0 for any k we are done

If not $g\left(\frac{k_1}{n}\right)$ and $g\left(\frac{k_2}{n}\right)$ must have opposite signs for some k_1 and k_2

This means there is a c in [k_1, k_2] such that g(c) = 0

i.e. there is a c such that f(c+1) - f(c) = 0 and the result follows