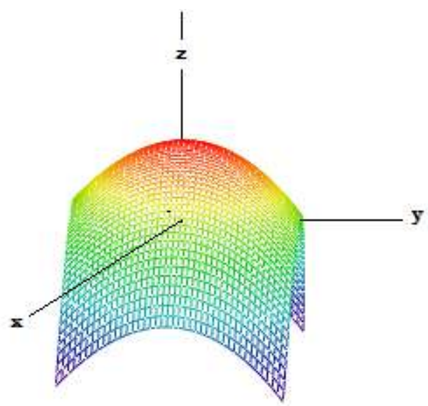
Classification of Critical Points - Contour Diagrams and Gradient Fields

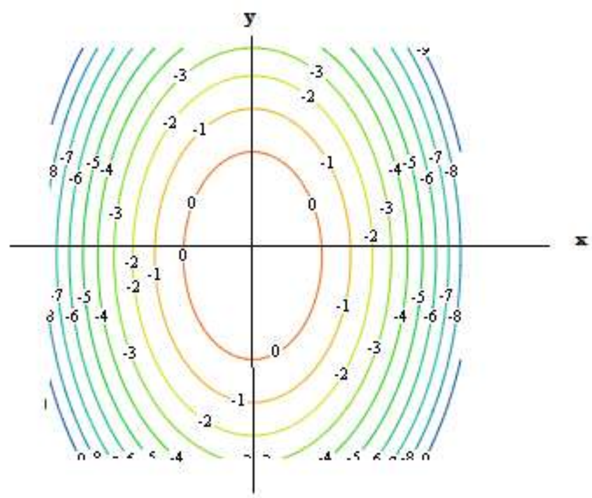
As we saw in the lecture on locating the critical points of a function of 2 variables there were three possibilities. A critical point could be a local maximum, a local minimum, or a saddle point.

There are 3 ways of classifying critical points. We consider 2 of those methods in this discussion

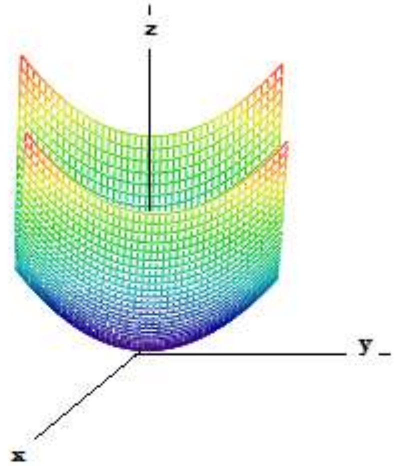
1. Using the contour diagram

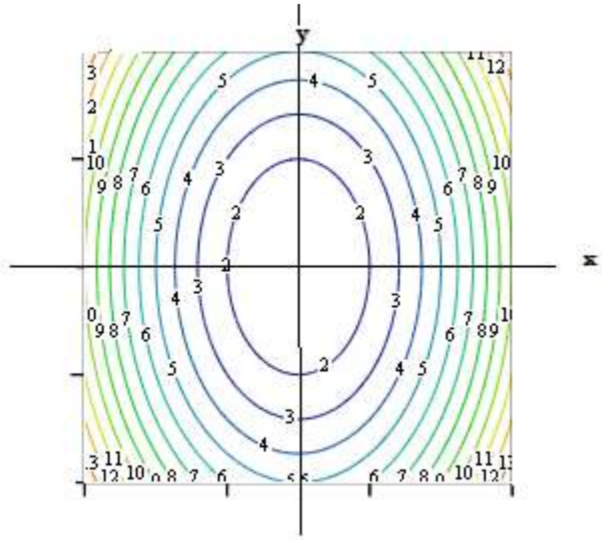
a. Local Maxima: In the contour diagram, locally, the critical point is the center of the contour and all contours increase as we move toward the critical point.





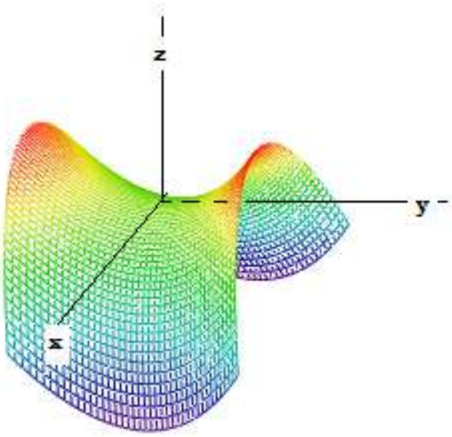
b. Local Minima: In the contour diagram, locally, the critical point is the center of the contour and all contours decrease as we move toward the critical point.

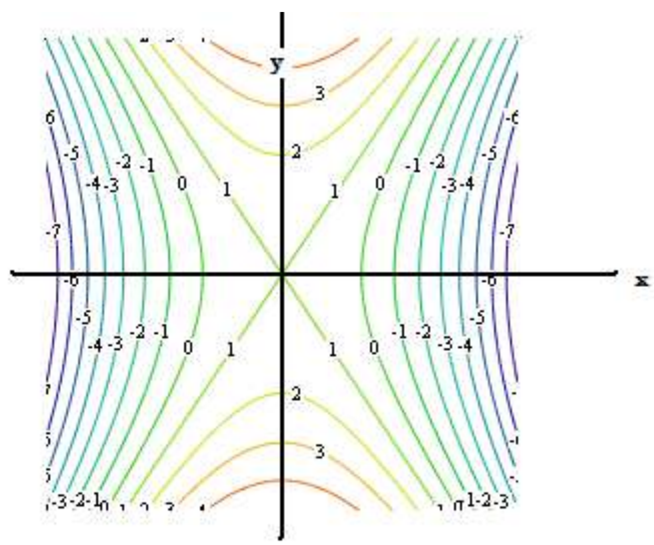




c. Saddle points: With a saddle point critical points are identified as the intersection of 2 contour lines.

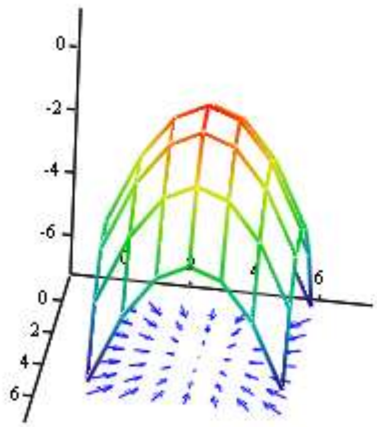
Moving in one direction the contours increase and in basically a perpendicular direction (not necessarily exactly perpendicular) the contours decrease. In the example below as we move left or right (in the $\pm$x direction) of the critical point the contours decrease but as we move up or down (in the $\pm$y direction) from the critical point the contours decrease.

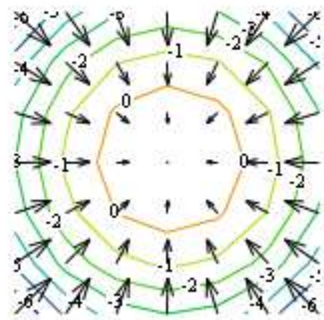




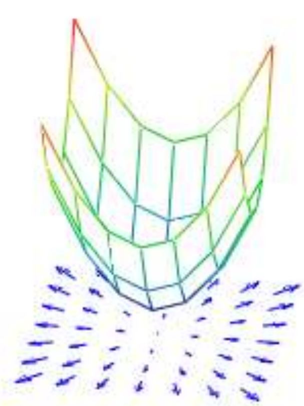
2. Using Gradient Fields - A gradient field is simply obtained by plotting several gradient vectors in the domain.

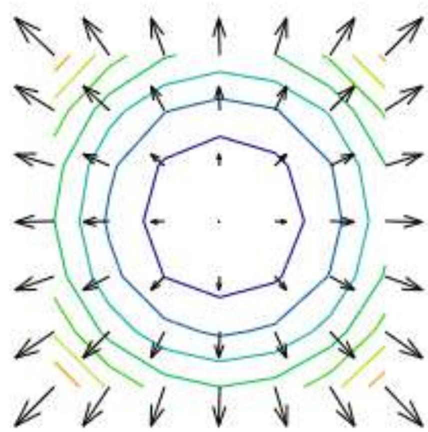
a. Local Maximum: If a critical point is a local maximum then all gradients in the neighborhood of that critical point are directed toward the critical point since in the neighborhood of a local maximum the function increases as we move toward the critical point.



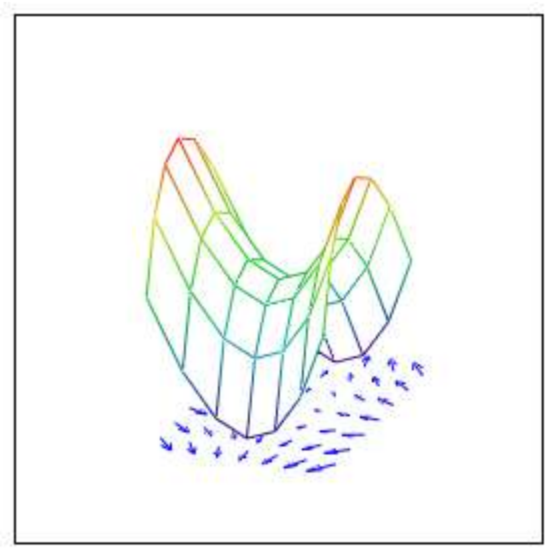


b. Local Minimum If a critical point is a local minimum then all gradients in the neighborhood of that critical point are directed away from the critical point since in the neighborhood of a local minimum the function decreases as we move toward the critical point.

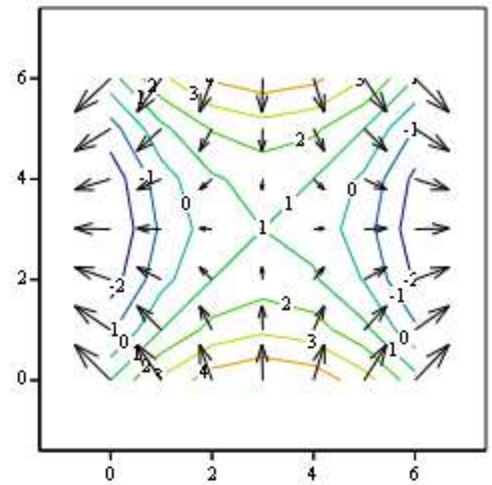




b. Saddle Point If a critical point is a saddle point then from one direction the gradients point toward the critical point but again from a basically perpendicular direction all gradients point away from the critical point.



(X,Y),N



N,(X,Y)

Example

Let's consider the example f(x,y) = $x^3 + y^3 - 3x - 3y$.

Then $\partial f / \partial x = 3x^2 - 3 = 0$ for $x = \pm 1$

$\partial f / \partial y = 3y^2 - 3 = 0$ for $y = \pm 1$

Therefore there are four critical points (1,1),(1,-1),(-1,1),(-1,-1).

For those of you who have Mathcad I've included all the formatting in the example below

$a := -2 \quad b := 2 \quad c := -2 \quad d := 2 \quad \Delta x := .1 \quad \Delta y := .1$

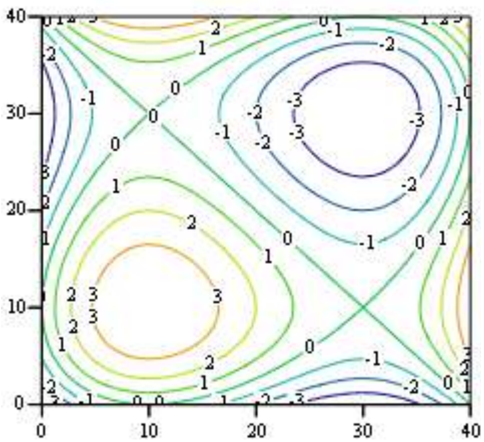
$i := 0 .. \frac{b-a}{\Delta x} \qquad j := 0 .. \frac{d-c}{\Delta y}$

$x_i := a + i \cdot \Delta x \quad y_j := c + j \cdot \Delta y$

$f(x,y) := x^3 + y^3 - 3 \cdot x - 3 \cdot y \qquad M_{i,j} := f\left(x_i, y_j\right)$

Using the Contour Diagram

Contour Diagram



M

To identify the x and y values of the points recall i corresponds to x and j corresponds to y. Therefore to identify the coordinates x_i and y_j type x_i = and y_j =.

$x_{10} = -1 \quad y_{30} = 1$ Saddle point : moving from (-1,1) vertically f increases but f decreases moving horizontally from (-1,1)

$x_{30} = 1 \qquad y_{30} = 1$ Minimum : f increases in all directions moving away from (1,1)

$x_{30} = 1 \qquad y_{10} = -1$ Saddle Point : moving from (1,-1) vertically f decreases but moving horizontally f increases from (1,-1)

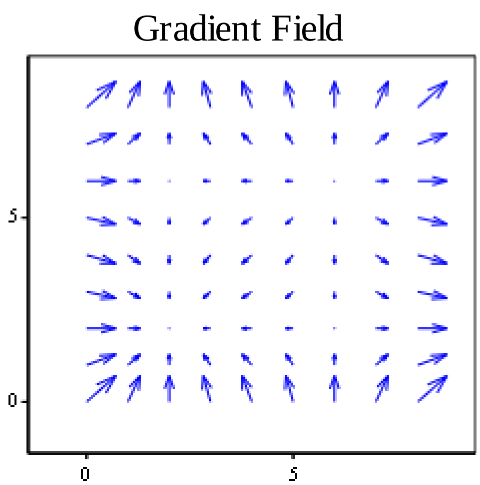
$x_{10} = -1 \quad y_{10} = -1$ Maximum : f decreases in all directions moving from (-1,-1)

Using The Gradient Field

$a := -2 \quad b := 2 \quad c := -2 \quad d := 2 \quad \Delta X := .5 \quad \Delta Y := .5$

$m := 0 .. \frac{b - a}{\Delta X} \quad n := 0 .. \frac{d - c}{\Delta Y} \quad x_m := a + m \cdot \Delta X \quad y_n := c + n \cdot \Delta Y$

$X_{m,n} := 3 \cdot (x_m)^2 - 3 \quad Y_{m,n} := 3 \cdot (y_n)^2 - 3$



(X, Y)

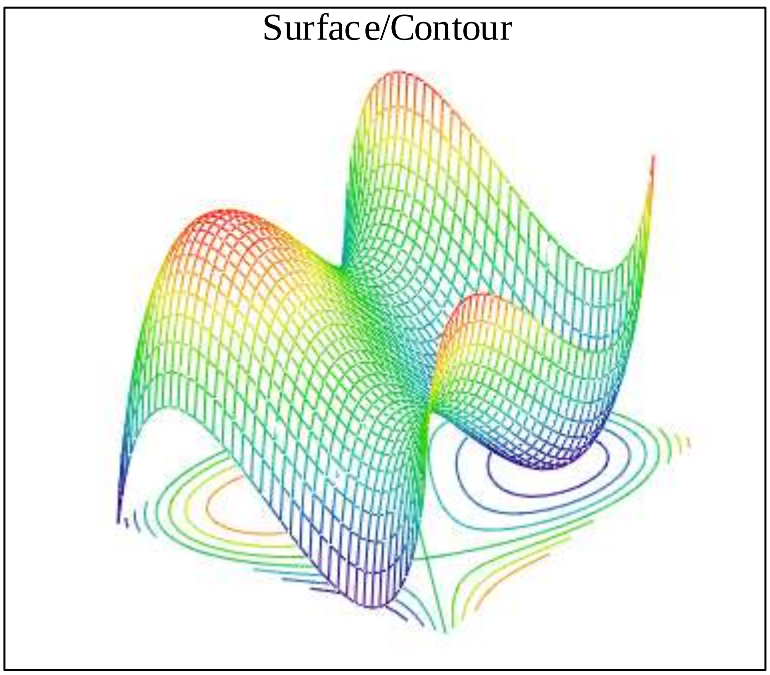
$x_2 = -1 \quad y_6 = 1$ Saddle point : Vertically the gradients point away from (-1,1), but horizontally the gradients point toward (-1,1)

$x_6 = 1 \quad y_2 = -1$ Saddle point : Vertically the gradients point toward from (1,-1), but horizontally the gradients point away from (1,-1)

$x_2 = -1 \quad y_2 = -1$ Maximum: all gradients point toward (-1,-1)

$x_6 = 1 \quad y_6 = 1$ Minimum: all gradients point away from (1,1)

Finally let's compare the surface with its contour diagram, and the contour diagram and the gradient field



M, M

$a := -2 \quad b := 2 \quad c := -2 \quad d := 2 \quad \Delta X := .4 \quad \Delta Y := .5$

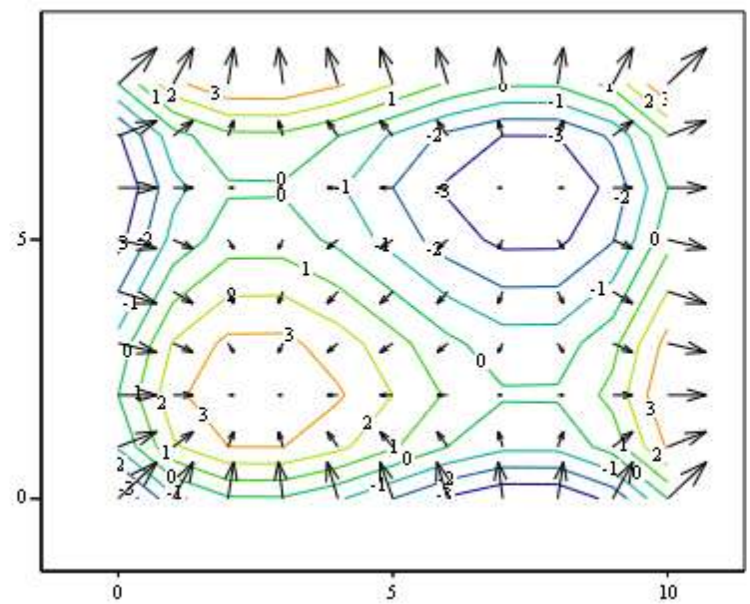
$i := 0 .. \frac{b - a}{\Delta X} \qquad j := 0 .. \frac{d - c}{\Delta Y}$

$m := 0 .. \frac{b - a}{\Delta X} \qquad n := 0 .. \frac{d - c}{\Delta Y} \qquad x_m := a + m \cdot \Delta X \qquad y_n := c + n \cdot \Delta Y$

$X_{m,n} := 3 \cdot (x_m)^2 - 3 \qquad Y_{m,n} := 3 \cdot (y_n)^2 - 3$

$f(x,y) := x^3 + y^3 - 3 \cdot x - 3 \cdot y \qquad R_{i,j} := f(x_i, y_j)$

Contour/Gradient



R, (X, Y)