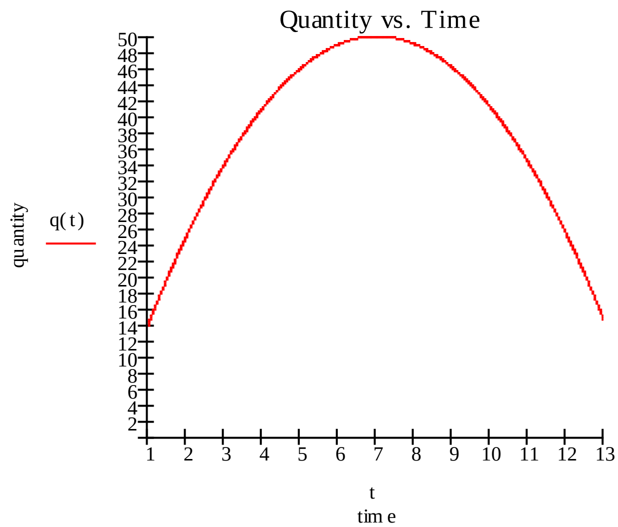
Chainrule Demonstration

Suppose we have a seasonal commodity in which the quantity of that commodity varies with time.
Further we would expect that the price is a function of the quantity.

So suppose our commodity is Asparagus and that the quantity as a function of time is given by : $q(t) := -(t-7)^2 + 50$ where t is in months. (to be precise here we define a month to be 1/12 of a year).

Note t =1 corresponds to Jan 1 and t =13 would correspond to Dec 31. The maximum is 50 which would be on July 1, t =7.




Further Suppose that the price of Asparagus is given by : $p(q) := 10 - 2 \cdot \ln(q)$


Note that the price varies from a maximum of $3.20 when the quantity is at a minimum of 14 and to a minimum of $2.15 when the quantity is a maximum of 50.

See the Animation price and quantity

You should have observed that starting at t =1 the quantity increases and the price moves down the price curve.

On July 1 the quantity begins to decrease and the the price reverses direction and moves up the price curve.

We know that the quantity is a function of time therefore $\frac{dq}{dt}$ is the rate at which the quantity of Asparagus is changing with respect to time in fact $\frac{dq}{dt}$ = -2( t - 7).

Further we know that $\frac{dp}{dq}$ is the rate at which the price is changing with respect to quantity in fact

$\frac{dp}{dq} = \frac{-2}{q}$ .

We then would expect that there is a way in which we can compute $\frac{dp}{dt}$ the rate at which the price is changing with respect to time. But How ?

This is what we want to explore in this discussion.

In our particular example p = p(q) and q = q(t) what is $\frac{dp}{dt}$ ?

Naively We might think that it is as simple as multiplying the two results i.e.

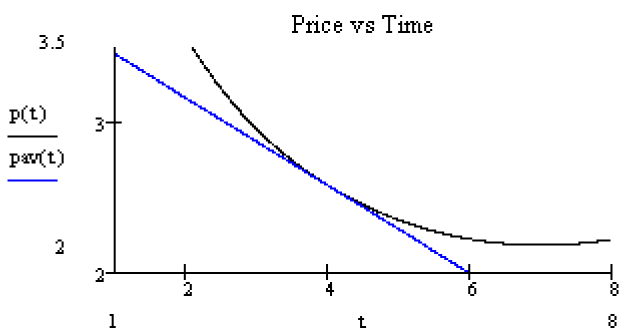
$$\frac{dp}{dt} = \left(\frac{dp}{dq}\cdot\frac{dq}{dt}\right) = \left[\frac{-2}{q}\cdot(-2\cdot(t-7))\right] = \frac{-2}{-(t-7)^2+50}\cdot(-2\cdot(t-7))$$

In particular at t=4 $\frac{dp}{dt} = \frac{-2}{-(4-7)^2+50}\cdot(-2\cdot(4-7))$ = -.293

See the Animation Rates of Change

Now let's consider this problem from another perspective.

Since p(q) = $10-2\cdot\ln(q)$ and q(t) = $-(t-7)^2+50$ it follows p(t) = 10 - 2ln( $-(t-7)^2+50$ ). Using the secant line approximation with h= .01




$$\frac{\Delta p}{\Delta t} = \frac{p(4+h)-p(4)}{h} = -0.293$$

This is the same result as our naïve guess!!!

See the Animation Average Rate of Change

What does this example show ?

In particular if p = p(q) and q = q(t) then $\frac{dp}{dt} = \frac{dp}{dq}\frac{dq}{dt}$

Now let's try and explain this formally without having to resort to numerical results

Note $\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}$

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Also as $\Delta t$ goes to 0 $\Delta q$ goes to 0. See Animation 4

Therefore $\frac{dp}{dt} = \lim_{\Delta t \to 0} \left(\frac{\Delta p}{\Delta q}\cdot\frac{\Delta q}{\Delta t}\right) = \lim_{\Delta q \to 0} \frac{\Delta p}{\Delta q} \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dp}{dq}\frac{dq}{dt}$ that is $\frac{dp}{dt} = \frac{dp}{dq}\frac{dq}{dt}$.

This is the chain rule.

In general if f = f(u) and u = u(x) then $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$.

Examples Find f '(x) for the following

1. $f(x) = \sin(x^2 + 2x+1)$

We can write this as f(u) = sin(u) where $u = x^2 + 2x+1$

Then $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$. $= \cos(u)(2x+1) = \cos(x^2 + 2x+1)(2x+1)$

2. f(x) = ln[ cos(x)]

Then we can write f(u) = ln(u) where u = cos(x)

$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{1}{u}\cdot(-\sin(x)) = \frac{1}{\cos(x)}\cdot(-\sin(x)) = -\tan(x)$

3. f(x) = $e^{-x \cdot \cos(x)}$

Then we can write f(u) = $e^{u}$ where u = -xcos(x)

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = e^{u} \cdot (-\cos(x) + x \cdot \sin(x)) = e^{-x \cdot \cos(x)} \cdot (-\cos(x) + x \cdot \sin(x))$$

Of course with a little practice you'll be able to take an outside to inside approach without formally defining u but using a little patience at first will pay off in the long run.