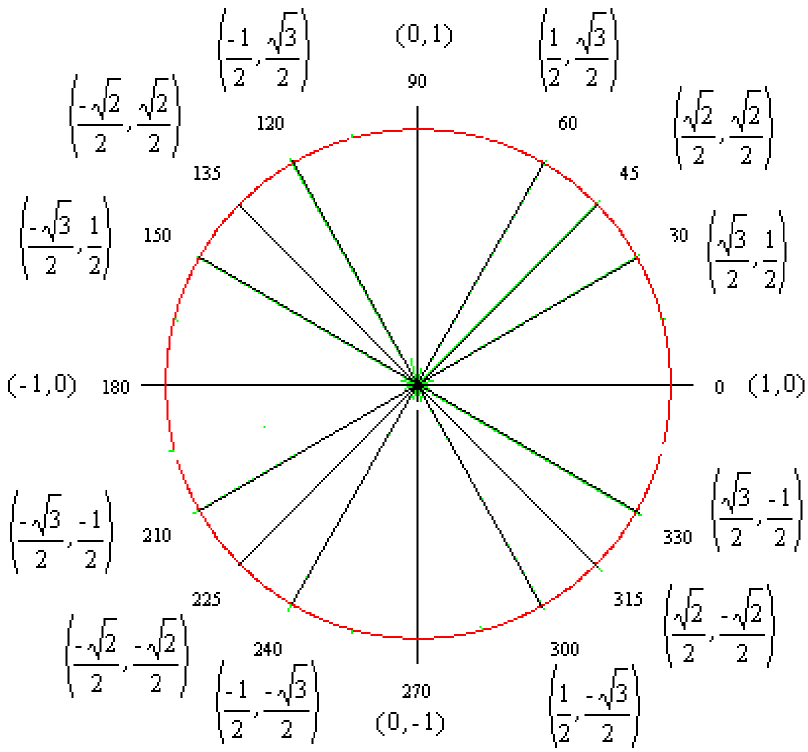
Trigonometry



Unit Circle
Note: Each ordered pair is$(\cos(\theta),\sin(\theta))$

Double Angle Identities

$\sin(2A) = 2\sin(A)\cos(A)$

$\cos(2A) = \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1= 1 - 2\sin^2(A)$

$$\tan(2A)=\frac{2\tan(A)}{1-\tan^2(A)}$$

Half Angle Identities

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}}$$

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$$

$$\tan\left(\frac{A}{2}\right)=\sqrt{\frac{1-\cos(A)}{1+\cos(A)}} = \frac{\sin(A)}{1+\cos(A)} = \frac{1-\cos(A)}{\sin(A)}$$

Identities For the Sum of 2 angles

cos(A + B) = cos(A)cos(B) - sin(A)sin(B)

cos(A - B) = cos(A)cos(B) + sin(A)sin(B)

sin(A + B) = sin(A)cos(B) + sin(B)cos(A)

sin(A - B) = sin(A)cos(B) - sin(B)cos(A)

$$\tan(A+B)=\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

$$\tan(A-B)=\frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

Pythagorean Identities

$\cos^2(\theta)+\sin^2(\theta)=1$ $\quad$ $\tan^2(\theta)+1=\sec^2(\theta)$ $\quad$ $\cot^2(\theta)+1=\csc^2(\theta)$

Basic Differentiation Formulas

Note : u is assumed to be a differentiable function of x u(x)

1. **Power Rule** $\frac{dx^n}{dx}=n\cdot x^{n-1}$

Chain Rule $\frac{du^n}{dx}=n\cdot u^{n-1}\cdot\frac{du}{dx}$

2. **Exponential Rules**

a. $\frac{de^x}{dx}=e^x$

b. Chain Rule $\frac{de^u}{dx}=e^u\cdot\frac{du}{dx}$

c. $\frac{da^x}{dx}=\ln(a)\cdot a^x$

d. Chain Rule $\frac{da^u}{dx}=\ln(a)\cdot a^u\frac{du}{dx}$

3. **Natural Logarithm**

$\frac{d\ln(x)}{dx}=\frac{1}{x}$ Chain Rule $\frac{d\ln(u)}{dx}=\frac{1}{u}\frac{du}{dx}$

4. **Trig Functions** Chain Rule

a. $\frac{d\sin(x)}{dx}=\cos(x)$ $\frac{d\sin(u)}{dx}=\cos(u)\frac{du}{dx}$

b. $\frac{d\cos(x)}{dx}=-\sin(x)$ $\frac{d\cos(u)}{dx}=-\sin(u)\frac{du}{dx}$

c. $\frac{d\tan(x)}{dx}=\sec^2(x)$ $\frac{d\tan(u)}{dx}=\sec^2(u)\frac{du}{dx}$

d. $\frac{d\cot(x)}{dx}=-\csc^2(x)$ $\frac{d\cot(u)}{dx}=-\csc^2(u)\frac{du}{dx}$

e. $\frac{d\sec(x)}{dx}=(\sec(x))\cdot\tan(x)$ $\frac{d\sec(u)}{dx}=(\sec(u))\cdot\tan(u)\frac{du}{dx}$

f. $\frac{d\csc(x)}{dx}=-\csc(x)\cot(x)$ $\frac{d\csc(u)}{dx}=-\csc(u)\cot(u)\frac{du}{dx}$

5. **Inverse Trig Functions** Chain Rule

a. $\frac{d\sin^{-1}(x)}{dx}=\frac{1}{\sqrt{1-x^2}}$ $\frac{d\sin^{-1}(u)}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

b. $\frac{d\cos^{-1}(x)}{dx}=\frac{-1}{\sqrt{1-x^2}}$ $\frac{d\cos^{-1}(u)}{dx}=\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$

c $\frac{d\tan^{-1}(x)}{dx}=\frac{1}{1+x^2}$ $\frac{d\tan^{-1}(u)}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$

d. $\frac{d\cot^{-1}(x)}{dx}=\frac{-1}{1+x^2}$ $\frac{d\cot^{-1}(u)}{dx}=\frac{-1}{1+u^2}\frac{du}{dx}$

e. $\frac{d\sec^{-1}(x)}{dx}=\frac{1}{x\cdot\sqrt{x^2-1}}$ $\frac{d\sec^{-1}(u)}{dx}=\frac{1}{u\cdot\sqrt{u^2-1}}\frac{du}{dx}$

f. $\frac{d\csc^{-1}(x)}{dx}=\frac{-1}{x\cdot\sqrt{x^2-1}}$ $\frac{d\csc^{-1}(u)}{dx}=\frac{-1}{u\cdot\sqrt{u^2-1}}\frac{du}{dx}$

Basic Integration Formulas

1. Power Rule

a. $n \neq -1$ $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

b. if n = -1 $\int \frac{1}{x} dx = \ln(|x|) + C$

2. Exponential Formulas

a. $\int e^x dx = e^x + C$

b. $\int a^x dx = \frac{a^x}{\ln(a)} + C$

3. Trig Formulas

a. $\int \sin(x) dx = -\cos(x) + C$

b. $\int \cos(x) dx = \sin(x) + C$

c. $\int \sec^2(x) dx = \tan(x) + C$

d. $\int \sec(x) \tan(x) dx = \sec(x) + C$

e. $\int \csc^2(x) dx = -\cot(x) + C$

f. $\int \csc(x) \cot(x) dx = -\csc(x) + C$

g. $\int \tan(x) dx = -\ln|\cos(x)| + C$

j. $\int \cot(x) dx = \ln|\sin(x)| + C$

4. Integrals involving Inverse Trig Functions

a. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$

b. $\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1}(x) + C$

c. $\int \frac{1}{x \cdot \sqrt{x^2-1}} dx = \sec^{-1}(x) + C$

d. $\int \frac{-1}{x \cdot \sqrt{x^2-1}} dx = \csc^{-1}(x) + C$

e. $\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + C$

f. $\int \frac{-1}{1+x^2} dx = \cot^{-1}(x) + C$

h. $\int \sec(x) dx = \ln|\tan(x) + \sec(x)| + C$

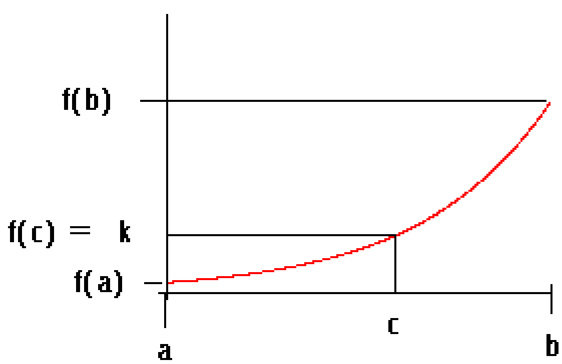
i. $\int \csc(x) dx = -(\ln|\cot(x) + \csc(x)|) + C$

Important Theorems

1. **Extreme Value Thm (EVT)**
Suppose f(x) is continuous on a closed and bounded interval.
Then f(x) has both a maximum and a minimum value on that interval.

Related Thm : if f(x) is also differentiable on that interval then an extreme value occurs at either an end point or at a stationary point.

2. **Intermediate Value Thm (IVT)**
Suppose f(x) is continuous on a closed and bounded interval [a,b].
Then if k is a number between f(a) and f(b) there is a number c in [a,b] such that f(c) = k.
See the picture below.



3. **Mean Value Theorem**

Suppose f(x) is continuous on[a,b] and differentiable on (a,b).
Then there is a number c in (a,b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.
This theorem is simply saying that there is some point c in(a,b) such that the instantaneous rate of change is equal to the average rate of change over the entire interval .

Or geometrically at some point c in (a,b) the slope of the tangent line is the same as the slope of the secant line through the endpoints.

This and its generalized form are probably the most important theorems in elementary Calculus; among many important results that will be derived using the MVT is the Fundamental Theorem of Calculus.

4. **Generalized MVT** also known as the Cauchy MVT or Extended MVT

Suppose f(x) and g(x) are continuous on [a,b] and continuous on (a,b) and further for any x in (a,b) $\frac{dg}{dx} \neq 0$.

Then there is a number c in (a,b) such that f ' (c)/ g'(c) = $\frac{f(b) - f(a)}{g(b) - g(a)}$.

Note if g(x) = x this reduces precisely to the MVT

5.**<u>Fundamental Theorem of Calculus</u>**

Suppose F ' (x) = f(x) i.e. F(x) is an antiderivative of f(x).

Then $\int_a^b f(x)\,dx = F(b) - F(a)$ or $\int_a^b \frac{dF}{dx}\,dx = F(b) - F(a)$

6. **<u>The Second Fundamental Theorem of Calculus</u>**

Suppose f(x) is continuous on (a,x) for all x in an open interval containing a

and Suppose F(x) = $\int_a^x f(t)\,dt$

Then F' (x) = f(x) .

The FTC and the 2d FTC establish that differentiation and integration are inverse operations.
The FTC establishes the Integral of the derivative of f(x) is f(x).
The 2d FTC establishes the derivative of the integral of f(x) is f(x).
6. **MVT for Integrals**

Suppose f is continuous on(a,b)

There is a number c in (a,b) such that $f(c) = \frac{1}{b - a} \cdot \int_a^b f(x)\,dx$.

This theorem is simply saying that there is point c where a function attains its average value. (Note this not true for a discrete set of data. For example if the average on a test is 85 it is not necessarily true that at least one student actually got an 85, however if your average speed is 85mph over a time interval at some point you were going 85mph.

Graphically we are saying there is a rectangle with dimensions(b-a)f(c) which is exactly equal to the area between f(x) and the x-axis for a $\leq$ x < <u>b</u>.

Computational Aspects of Limits at Infinity and Infinite Limits.

Before we get into the details let's consider what may seem some rather strange mathematical devices:

1. $\frac{c}{0} = \pm\infty$ where c cannot be 0.

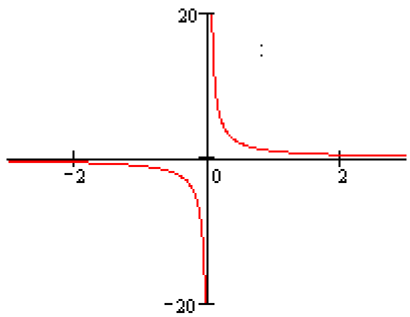
2. $\frac{c}{\infty} = 0$

3 $\frac{c}{-\infty} = 0.$

To understand why we make these definitions we need only understand the function f(x) = $\frac{1}{x}$:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to -\infty} \frac{1}{x} = \left(\frac{1}{-\infty} = 0\right) \qquad \lim_{x \to 0^-} \frac{1}{x} = \frac{1}{0} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0} = \infty \qquad \lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

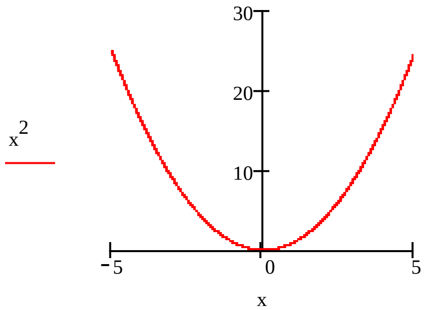
Also it is useful to understand limits at infinity and infinite limits of some of our elementary functions

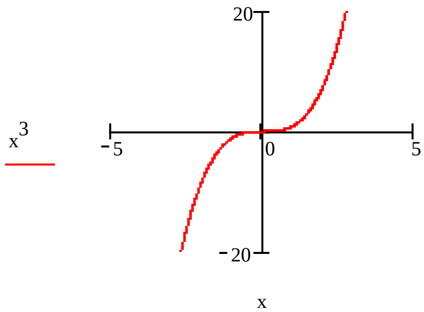
1. Power Functions

$$\lim_{x \to \infty} x^n = \infty$$

$\lim_{x \to -\infty} x^n = \infty$ if n is even

$\lim_{x \to -\infty} x^n = -\infty$ if n is odd





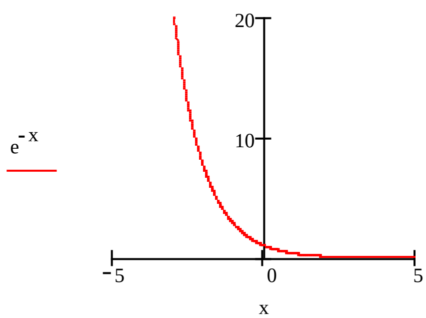
2. Exponentials

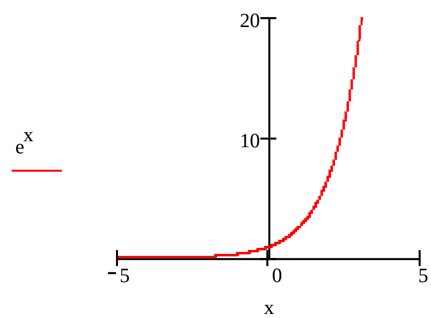
$$\lim_{x \to -\infty} e^{x} = 0$$

$$\lim_{x \to \infty} e^{x} = \infty$$

$$\lim_{x \to \infty} e^{-x} = 0$$

$$\lim_{x \to -\infty} e^{-x} = \infty$$

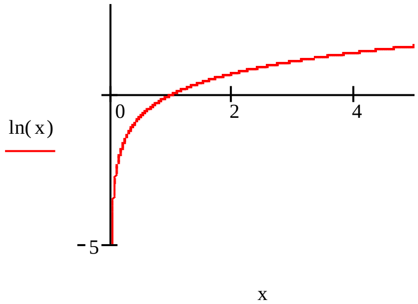




3. The Natural Logarithm

$$\lim_{x \to 0^{+}} \ln(x) = -\infty$$

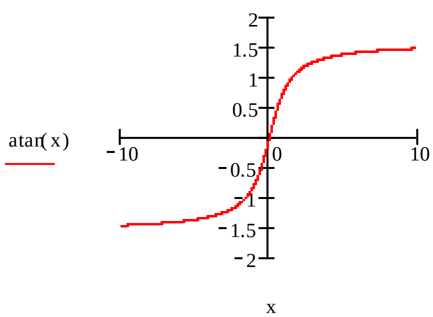
$$\lim_{x \to \infty} \ln(x) = \infty$$



4. Tan $^{-1}$ (x)

$$\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$$

$$\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}$$



<u>Important Point Not to Miss</u>

<u>The limits at infinity and negative are the HORIZONTAL ASYMPTOTES OF THE FUNCTION.</u>

<u>Note the Arctangent Function has a different Asymptote at negative infinity than it has at infinity.</u>

<u>We call these half-assymptotes. (ok not really)</u>