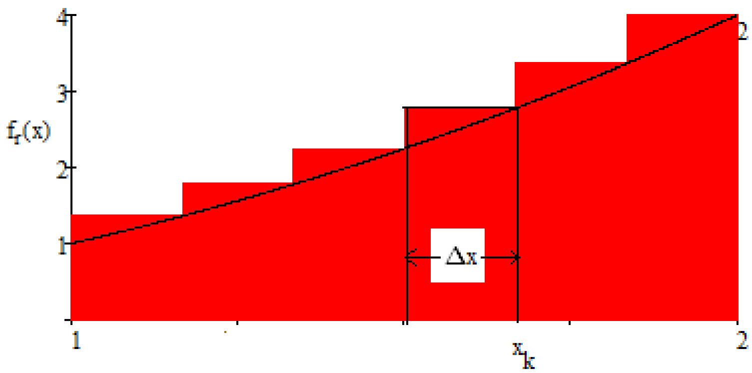
The Calculus of Polar Coordinates - Integrals

We have seen that in rectangular coordinates to compute the Area under a curve we partition

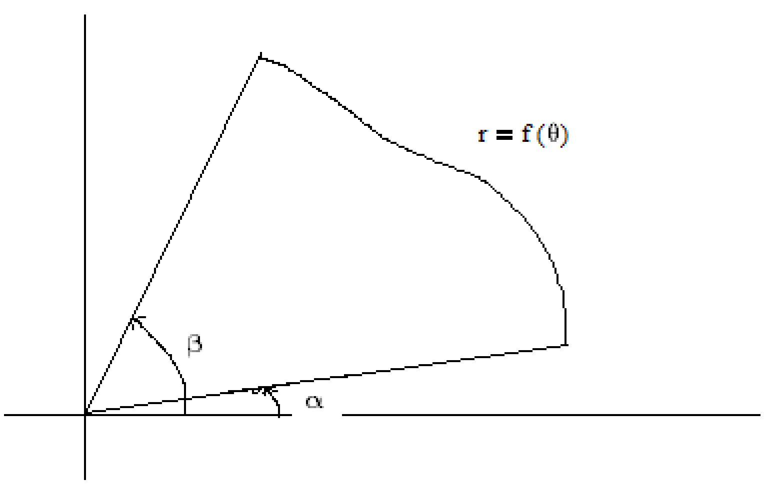
the x axis into a large number of subintervals. We then approximate the area on each subinterval by a rectangle.

We then sum the areas and take the limit as the number of subintervals goes to ∞.

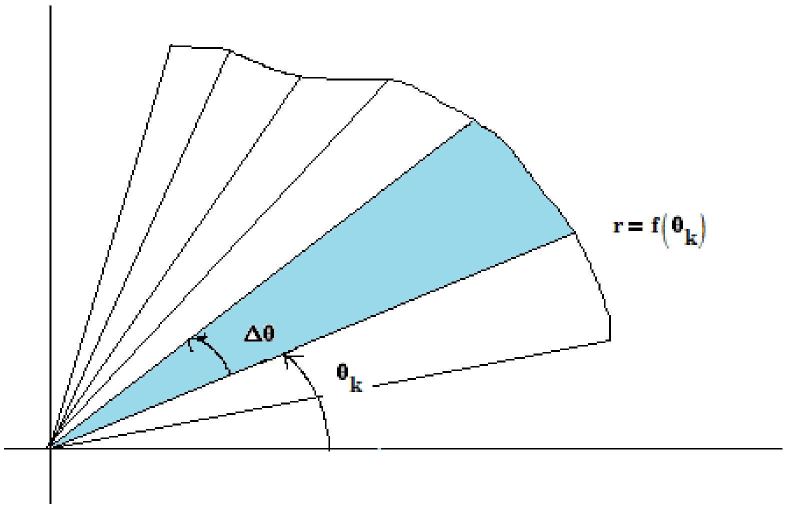


$$A = \lim_{n \to \infty} \sum_{k=1}^{n} \left(f\left(x_k\right) \cdot \Delta x \right)$$

Suppose we have a curve in polar coordinates



We start by partitioning this region according to θ = constant which are lines emanating form the origin.



Instead of using a rectangle to approximate this area we use the area of a circular sector.

Recall the area of a circular sector is $\mathbf{A} = \frac{1}{2}\cdot\mathbf{r}^2\cdot\theta.$ For this one particular sector $\mathbf{r} = \mathbf{f}(\theta_{\mathbf{k}}).$

We have the area of this one sector $\Delta\mathbf{A} = \frac{1}{2}\cdot\mathbf{f}(\theta_{\mathbf{k}})^2\cdot\Delta\theta$

Then we define the area as $\mathbf{A} = \lim_{\mathbf{n}\to\infty}\sum_{\mathbf{k}=1}^{\mathbf{n}}\left(\frac{1}{2}\cdot\mathbf{f}(\theta_{\mathbf{k}})^2\cdot\Delta\theta\right) = \frac{1}{2}\cdot\int_{\alpha}^{\beta}\mathbf{f}(\theta)^2\,\mathbf{d}\theta$

See Animation 3

Note as n increases Δ θ decreases and the circular sectors become vary narrow. Compare this with Riemann Sums in rectangular coordinates where as n increases Δ x decreases and the rectangles narrow.

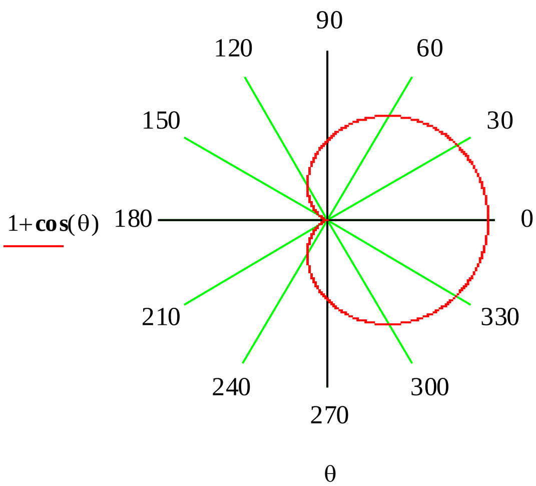
Because we have $\mathbf{f}(\theta)^2$ in the integrand often we end up with integrals of square of sin(x) and cos(x).

Recall

$$\int \sin(\mathbf{x})^2\,\mathbf{dx} \to \frac{\mathbf{x}}{2} - \frac{\sin(2\cdot\mathbf{x})}{4} \quad \text{or} \quad \frac{\mathbf{x}}{2} - \frac{\sin(\mathbf{x})\cdot\cos(\mathbf{x})}{2}$$

$$\int \cos(\mathbf{x})^2\,\mathbf{dx} \to \frac{\mathbf{x}}{2} + \frac{\sin(2\cdot\mathbf{x})}{4} \quad \text{or} \quad \frac{\mathbf{x}}{2} + \frac{\sin(\mathbf{x})\cdot\cos(\mathbf{x})}{2}$$

Example 1 Find the Area enclosed by the Cardioid r = 1 + cos(θ)



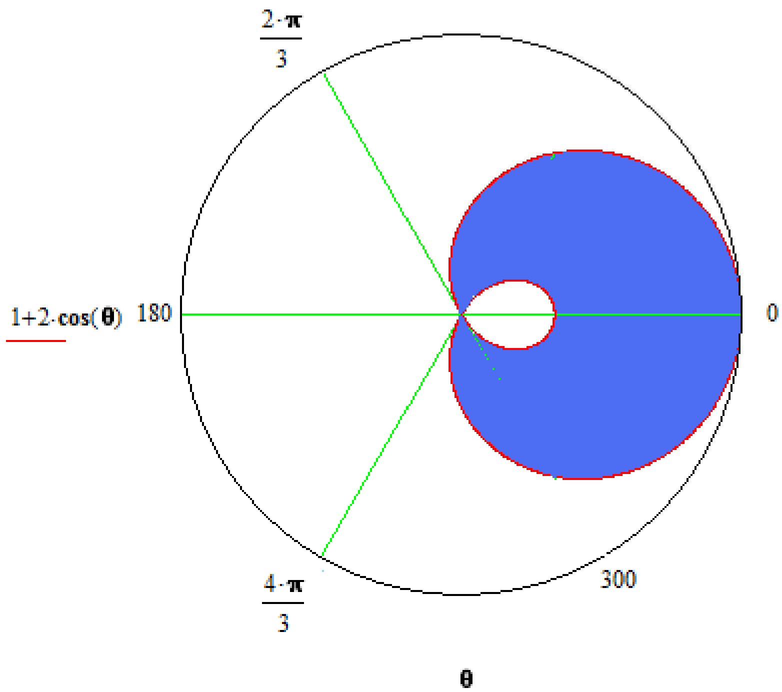
Here we can compute the upper half and double the result

$$\mathbf{A} = \frac{1}{2}\cdot\int_{\alpha}^{\beta} \mathbf{f}(\theta)^2 \, \mathbf{d}\theta = \int_0^{\pi} (1 + \mathbf{cos}(\theta))^2 \, \mathbf{d}\theta = \int_0^{\pi} \left(\mathbf{cos}(\theta)^2 + 2\cdot\mathbf{cos}(\theta) + 1\right) \mathbf{d}\theta$$

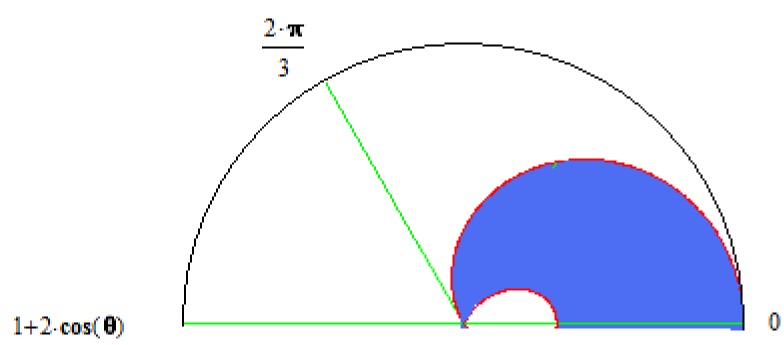
$$\int_0^{\pi} \left(\mathbf{cos}(\theta)^2 + 2\cdot\mathbf{cos}(\theta) + 1\right) \mathbf{d}\theta = \left(\frac{\theta}{2} - \frac{\mathbf{sin}(\theta)\cdot\mathbf{cos}(\theta)}{2} + 2\cdot\mathbf{sin}(\theta) + \theta\right)\cdot \Big|_0^{\pi} \quad = \frac{3\cdot\pi}{2}$$

Example 2

Find the area inside the outer loop but outside the inner loop of the limacon r = 1 + 2 cos(θ)



Note we can again calculate the Area and double the result

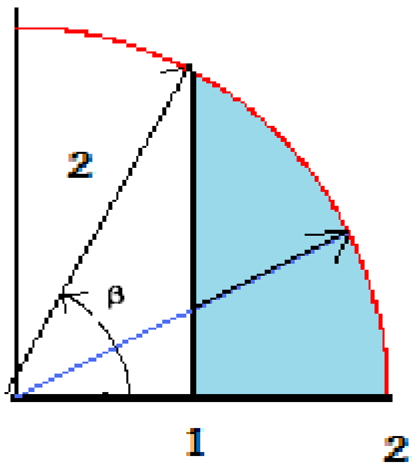


$$\mathbf{A} = \int_{0}^{2\cdot\frac{\pi}{3}} (1 + 2\cdot\cos(\theta))^2 \, d\theta - \int_{\frac{2\cdot\pi}{3}}^{\pi} (1 + 2\cdot\cos(\theta))^2 \, d\theta$$

$$\mathbf{A} = 2\cdot\pi + \frac{3\cdot\sqrt{3}}{2} - \left(\pi - \frac{3\cdot\sqrt{3}}{2}\right) = \pi + 3\cdot\sqrt{3}$$

Example 3

Find the area in the circle r = 2 to the right of the line x = 1 i the first quadrant



Here we have the area between 2 curves .

In the first curve r varies from 0 to the line x = 1 .   In the second r varies from 0 to the circle r = 2 as θ   varies from 0 to β .

We need   x =1 in polar form : x = rcos(θ) = 1

It follows $\mathbf{r} = \mathbf{sec}(\theta)$

To Calculate β   we have $\mathbf{cos}(\beta) = \frac{1}{2}$ which yields β   = π /3

$$\mathbf{A} = \frac{1}{2}\cdot\int_0^{\frac{\pi}{3}} \left[2^2 - \mathbf{sec}^2(\theta)\right] \mathbf{d}\theta = \frac{1}{2}\cdot\int_0^{\frac{\pi}{3}} \left[4 - \mathbf{sec}^2(\theta)\right] \mathbf{d}\theta = \frac{1}{2}\cdot(4\cdot\theta - \mathbf{tan}(\theta))\cdot \Bigg|_0^{\frac{\pi}{3}} = \frac{2\cdot\pi}{3} - \frac{\sqrt{3}}{2}$$

Of Course we could have taken the area of the circular sector - area of the triangle:

$$\frac{1}{2}\cdot 2^2\cdot\frac{\pi}{3} - \frac{1}{2}\cdot 1\cdot\left(\sqrt{2^2 - 1}\right) = \frac{2\cdot\pi}{3} - \frac{\sqrt{3}}{2}$$

However when we get to vector calculus and have to compute the mass of a region like this
we won't be able to appeal to simple geometry so it is instructive to set up this area as an integral.