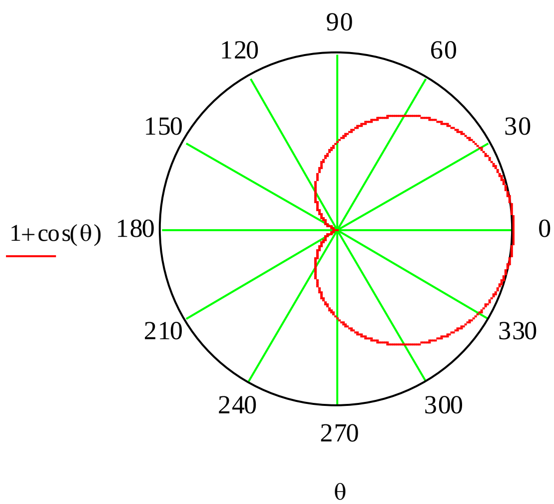
The Calculus of Polar Coordinates - Derivatives

In rectangular coordinates you've learned $\frac{dy}{dx}$ is the slope of the tangent line to a curve at a point. But what about $r = f(\theta)$? At first you might think $\frac{dr}{d\theta}$ is the slope of the tangent line to the curve but consider r = constant e.g. r = 1 which is of course a circle.

$\frac{dr}{d\theta} = 0$ at every point which is obviously not the slope of the tangent line. What does $\frac{dr}{d\theta}$ compute ? It is the rate at which the distance from the origin to the curve changes with respect to a change in θ . It makes sense for r = 1 then that $\frac{dr}{d\theta} = 0$ since the distance doesn't change as we move along a circle.

As another example consider the cardioid $r = 1 + \cos(\theta)$.



It follows $\frac{dr}{d\theta} = -\sin(\theta)$

Then it follows

$\frac{dr}{d\theta} < 0$ for $0 < \theta < \pi$ i.e. r decreases for $0 < \theta < \pi$.

and $\frac{dr}{d\theta} > 0$ for $\pi < \theta < 2\pi$ i.e. r increases for $\pi < \theta < 2\pi$.

See animation 1.

This brings us to the question - How do we calculate the slope of the tangent line to a polar curve?

The answer lies in parametric equations.

Recall if

$$x = x(t)$$
$$y = y(t)$$

Then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

In Polar coordinates:

$$x = r \cdot \cos(\theta)$$
$$y = r \cdot \sin(\theta)$$

But recall r = f(θ)

We have $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\frac{dr}{d\theta}\cdot\sin(\theta) + r\cdot\cos(\theta)}{\frac{dr}{d\theta}\cdot\cos(\theta) - r\cdot\sin(\theta)}$

the slope of the tangent line to a polar graph is

$$m_{tan} = \frac{\frac{dr}{d\theta}\cdot\sin(\theta) + r\cdot\cos(\theta)}{\frac{dr}{d\theta}\cdot\cos(\theta) - r\cdot\sin(\theta)}$$

For example for r = 1

$$m_{tan} = \frac{\frac{dr}{d\theta}\cdot\sin(\theta) + r\cdot\cos(\theta)}{\frac{dr}{d\theta}\cdot\cos(\theta) - r\cdot\sin(\theta)} = \frac{r\cdot\cos(\theta)}{-(r\cdot\sin(\theta))} = \frac{-x}{y}$$

Looking at this in rectangular coordinates

$$x^2 + y^2 = 1$$

Differentiating implicitly:

$2x + 2y\cdot\dfrac{dy}{dx} = 0$ We again obtain $\dfrac{dy}{dx} = \dfrac{-x}{y}$

Let's return to our cardioid : $1 + \cos(\theta)$

$$m_{tan} = \frac{\frac{dr}{d\theta}\cdot\sin(\theta) + r\cdot\cos(\theta)}{\frac{dr}{d\theta}\cdot\cos(\theta) - r\cdot\sin(\theta)} = \frac{-\sin(\theta)\sin(\theta) + (1+\cos(\theta))\cdot\cos(\theta)}{-\sin(\theta)\cos(\theta) - [(1+\cos(\theta))\cdot\sin(\theta)]} = \frac{2\cdot\cos^2(\theta) + \cos(\theta) - 1}{-2\sin(\theta)\cdot\cos(\theta) - \sin(\theta)}$$

We want to consider in particular where m_{tan} is either 0 or undefined to determine where

we have horizontal tangents, vertical tangents or cusps.

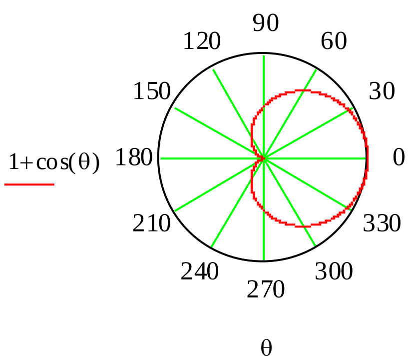
The numerator is 0 for $2\cdot\cos^2(\theta) + \cos(\theta) - 1 = 0$ which yields $[2\cdot(\cos(\theta) - 1)]\cdot(\cos(\theta + 1) = 0$

which yields θ = π , π /3, and 5π /3

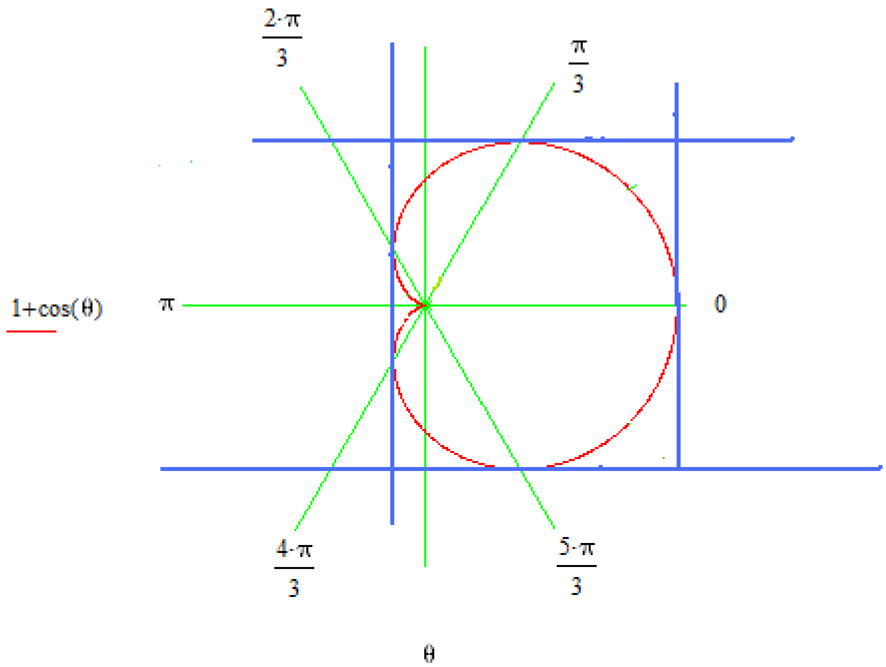
The denominator is 0 for $-2\sin(\theta)\cdot\cos(\theta) - \sin(\theta) = 0$ which yields $\sin(\theta) = 0$ or $\cos(\theta) = \dfrac{-1}{2}$

This yields θ = 0,π ,2π /3, and 4π /3

From the graph



We have horizontal tangents at θ = π /3 and 5π /3.
We have vertical tangents at θ = 0, 2π /3 and 4π /3
We have a cusp at θ = π
See Animation 2.



One final Note --How do we plot the tangent line to a polar graph?

Let's start with the tangent line in rectangular coordinates.

$y = m(x - x_0) + y_0$ where (x_0, y_0) is the point where we want the tangent line.

in polar coordinates (x_0, y_0) is replaced by $(r(\theta_0), \theta_0)$ is a specific point and

and m is replaced by $m_{tan} = \dfrac{\frac{dr}{d\theta}\cdot\sin(\theta_0) + r\cdot\cos(\theta_0)}{\frac{dr}{d\theta}\cdot\cos(\theta_0) - r\cdot\sin(\theta_0)}$

$$r\cdot\sin(\theta) = m_{tan}\cdot\left(r\cdot\cos(\theta) - r\left(\theta_0\right)\cdot\cos\left(\theta_0\right)\right) + r\left(\theta_0\right)\cdot\sin\left(\theta_0\right)$$

Solving for r :

$$r\cdot\left(\sin(\theta) - m_{tan}\cdot\cos(\theta)\right) = -\left[m_{tan}\cdot\left(r\left(\theta_0\right)\cdot\cos\left(\theta_0\right)\right)\right] + r\left(\theta_0\right)\cdot\sin\left(\theta_0\right)$$

$$r = \frac{-\left[m_{tan}\cdot\left(r\left(\theta_0\right)\cdot\cos\left(\theta_0\right)\right)\right] + r\left(\theta_0\right)\cdot\sin\left(\theta_0\right)}{\left(\sin(\theta) - m_{tan}\cdot\cos(\theta)\right)}$$

where $m_{tan} = \dfrac{\dfrac{dr}{d\theta}\cdot\sin\left(\theta_0\right) + r\cdot\cos\left(\theta_0\right)}{\dfrac{dr}{d\theta}\cdot\cos\left(\theta_0\right) - r\cdot\sin\left(\theta_0\right)}$

For example with our Cardioid the polar form of the tangent line at π /6 we have:

$$m_{tan} = \frac{2\cdot\cos^2(\theta) + \cos(\theta) - 1}{-2\sin(\theta)\cdot\cos(\theta) - \sin(\theta)}$$

$$m_{tan} = \frac{2\cdot\cos^2\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) - 1}{-2\sin\left(\frac{\pi}{6}\right)\cdot\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)} = -1$$

$$r = \frac{\left[-\left[\left(1 + \cos\left(\frac{\pi}{6}\right)\right)\cdot\cos\left(\frac{\pi}{6}\right)\right] + \left(1 + \cos\left(\frac{\pi}{6}\right)\right)\cdot\sin\left(\frac{\pi}{6}\right)\right]}{[\sin(\theta) - (-1)\cdot\cos(\theta)]}$$

$$r = \frac{-.683}{\sin(\theta) + \cos(\theta)}$$