Expressing the Acceleration in Terms of the unit Tangent and the Unit Normal

We have seen that given $\vec{r}(t) = x(t)\cdot\vec{i} + y(t)\cdot\vec{j}$ then the velocity vector is $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\cdot\vec{i} + \frac{dy}{dt}\cdot\vec{j}$

and the velocity vector is tangent to the curve at each point along the trajectory.

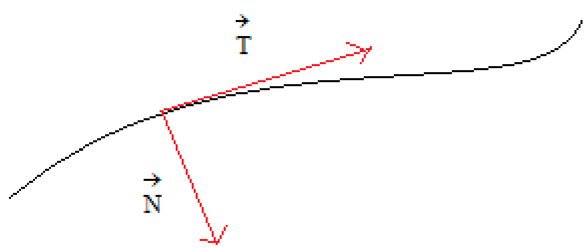
Further the acceleration vector is $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\cdot\vec{i} + \frac{d^2y}{dt^2}\cdot\vec{j}$.

However, there are 2 ways in which a particle can accelerate: either due to a change in speed or a change in direction. In this form of the acceleration we have we can't really see this.

We want to express the acceleration in terms of a tangential component a_T and a normal component a_N.

The tangential component will tell us the change in speed and the normal component will give us the change in direction. In this discussion we concentrate on finding $\vec{T}$ and $\vec{N}$ the unit tangent and unit normal vectors.

In subsequent discussions we will concentrate on finding find a_T and a_N .



Let $\vec{T}$ be the unit tangent . Since we know the velocity is tangent to the curve it follows:

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$$

We claim the unit normal is $\vec{N} = \dfrac{\overrightarrow{\frac{dT}{dt}}}{\left|\overrightarrow{\frac{dT}{dt}}\right|}$. It is obvious it is a unit vector so all that remains to be shown is

that $\vec{T}$ is perpendicular to $\overrightarrow{\frac{dT}{dt}}$ i.e. $\vec{T}\cdot\overrightarrow{\frac{dT}{dt}} = 0$.

Thm

Let $\overrightarrow{w(t)}$ be any vector valued function.

If $\left|\overrightarrow{w(t)}\right| = \text{constant}$ then $\overrightarrow{w(t)}\cdot\overrightarrow{\frac{dw}{dt}} = 0$

Pf:

$$\frac{d}{dt}\left(\vec{w}\cdot\vec{w}\right) = \overrightarrow{\frac{dw}{dt}}\cdot\vec{w} + \vec{w}\cdot\overrightarrow{\frac{dw}{dt}} = 2\cdot\vec{w}\cdot\overrightarrow{\frac{dw}{dt}}$$

But $\frac{d}{dt}\left(\vec{w}\cdot\vec{w}\right) = \frac{d}{dt}\left[\left|\left(\vec{w}\right)\right|\right]^2$

and $\left[\left|\left(\vec{w}\right)\right|\right]^2$ is a constant

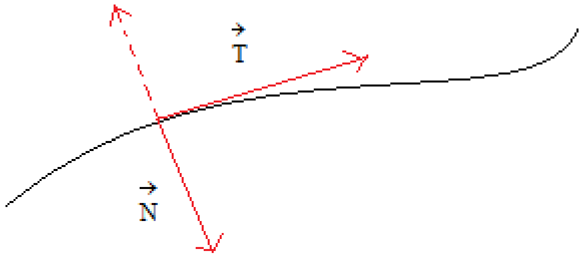
Therefore $\frac{d}{dt}\left(\vec{w}\cdot\vec{w}\right) = 0 = 2\cdot\vec{w}\cdot\overrightarrow{\frac{dw}{dt}}$

It follows $\vec{T}\cdot\overrightarrow{\frac{dT}{dt}} = 0$ is a corollary as $\left|\vec{T}\right| = 1$

Summary

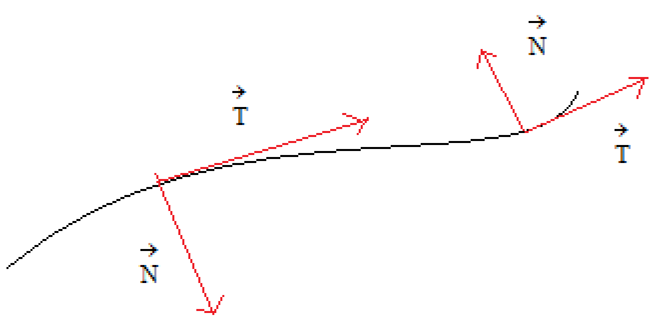
The unit tangent is $\vec{T} = \dfrac{\overrightarrow{\frac{dr}{dt}}}{\left|\overrightarrow{\frac{dr}{dt}}\right|}$ and the unit normal is $\vec{N} = \dfrac{\overrightarrow{\frac{dT}{dt}}}{\left|\overrightarrow{\frac{dT}{dt}}\right|}$

Now we have another problem. In 2 space there are actually 2 possibilities for $\vec{N}$ an inward and an outward normal . (In 3space there are an infinity of possibilities)



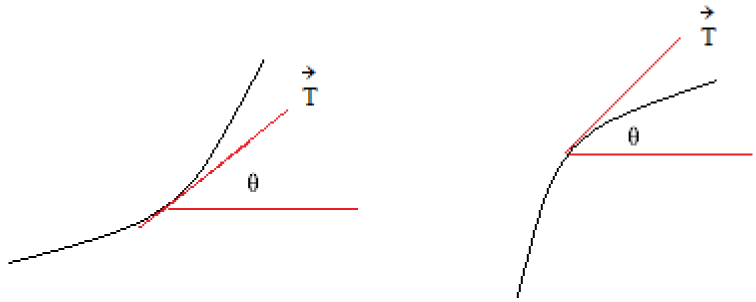
We want $\vec{N}$ to be the vector pointing inward and we call this the inward unit normal.

Does $\vec{N} = \dfrac{\dfrac{d\vec{T}}{dt}}{\left|\dfrac{d\vec{T}}{dt}\right|}$ satisfy this at all points ?



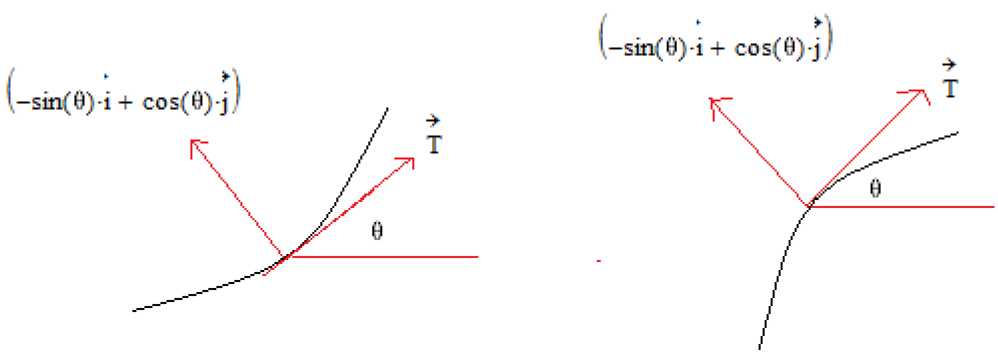
Recall if we have any unit vector $\vec{u}$ then it can be written $\vec{u} = \cos(\theta)\cdot\vec{i} + \sin(\theta)\cdot\vec{j}$

where θ is the angle between $\vec{u}$ and the horizontal measured counterclockwise from the horizontal.



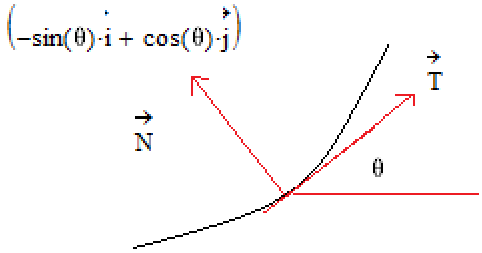
$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{d\theta}\cdot\frac{d\theta}{dt} = \left(-\sin(\theta)\cdot\vec{i} + \cos(\theta)\cdot\vec{j}\right)\cdot\frac{d\theta}{dt}$$

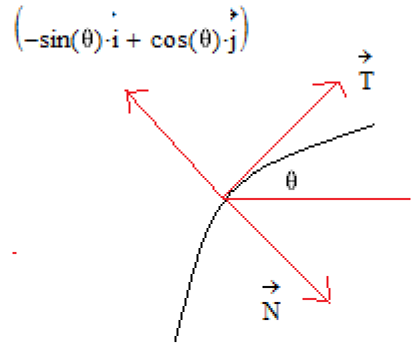
Note $\left(-\sin(\theta)\cdot\vec{i} + \cos(\theta)\cdot\vec{j}\right)$ is a 90^0 counter clockwise rotation of $\vec{T}$



Note in the diagram on the left $\frac{d\theta}{dt} > 0$ so $\frac{d\vec{T}}{dt}$ is in the same direction as $\left(-\sin(\theta)\cdot\vec{i} + \cos(\theta)\cdot\vec{j}\right)$ pointing inward along the curve.

Note in the diagram on the right $\frac{d\theta}{dt} < 0$ so $\frac{d\vec{T}}{dt}$ is oppositely directed to $\left(-\sin(\theta)\cdot\vec{i} + \cos(\theta)\cdot\vec{j}\right)$ and again pointing inward along the curve.





See Animation 1

Example

$$\vec{r}(t) = t\cdot\vec{i} + t^2\cdot\vec{j}$$

$$\vec{T}(t) = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{1}{\sqrt{1+4\cdot t^2}}\cdot\left(\vec{i} + 2\cdot t\cdot\vec{j}\right)$$

The Calculation of $\vec{N}$ is a little more difficult

$$\frac{d\vec{T}}{dt} = \frac{-4\cdot t}{\left(1+4\cdot t^2\right)^{\frac{3}{2}}}\cdot\left(\vec{i} + 2t\cdot\vec{j}\right) + \frac{1}{\sqrt{1+4\cdot t^2}}\cdot\left(2\vec{j}\right) = \frac{1}{\left(1+4\cdot t^2\right)^{\frac{3}{2}}}\cdot\left(-4\cdot t\cdot\vec{i} + 2\cdot\vec{j}\right)$$

$$\left|\frac{d\vec{T}}{dt}\right| = \left|\frac{1}{\left(1+4\cdot t^2\right)^{\frac{3}{2}}}\cdot\left(-4\cdot t\cdot\vec{i} + 2\cdot\vec{j}\right)\right| = \frac{1}{\left(1+4\cdot t^2\right)^{\frac{3}{2}}}\cdot\left(\sqrt{16t^2+4}\right)$$

$$\vec{N} = \frac{\dfrac{d\vec{T}}{dt}}{\left|\dfrac{d\vec{T}}{dt}\right|} = \frac{\dfrac{1}{\left(1+4\cdot t^2\right)^{\frac{3}{2}}}}{\dfrac{1}{\left(1+4\cdot t^2\right)^{\frac{3}{2}}}\cdot\left(\sqrt{16\cdot t^2+4}\right)}\left(-4\cdot t\cdot\vec{i}+2\cdot\vec{j}\right) = \frac{1}{2\cdot\sqrt{4\cdot t^2+1}}\cdot\left(-4\cdot t\cdot\vec{i}+2\cdot\vec{j}\right)$$

See Animation 2