The Second Partial Derivative Test

We will be using δ to denote the partial derivative operator.

In our previous lecture we saw how to classify extrema using contour diagrams and gradient fields.

Here we develop a method for classifying critical points without using graphical techniques.

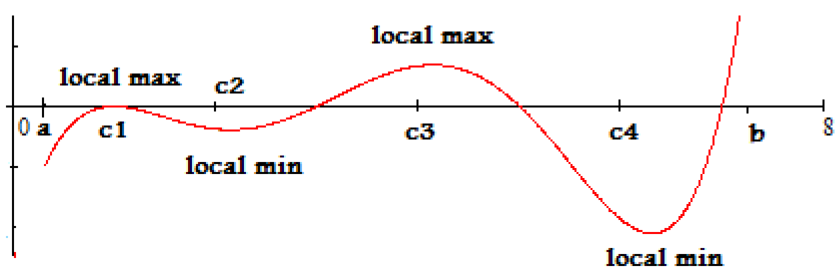
Recall from Calculus 1 you learned the Second Derivative Test. If f(x) has a critical point at x = c

Then :

1. If f '' (c) > 0 f(x) is concave up at x = c and therefore has a minimum at x = c.

2. If f '' (c) < 0 f(x) is concave down at x = c and therefore has a minimum at x = c.

See the diagram below



For functions of 2 variables we analogously have a Second Partial Derivative Test:

Let $\left(x_0, y_0\right)$ be a critical point. and let $D = \frac{\delta^2 f}{\delta x^2}\left(x_0, y_0\right) \cdot \left[\frac{\delta^2 f}{\delta y^2}\left(x_0, y_0\right)\right] - \left[\frac{\delta^2 f}{\delta y \delta x}\left(x_0, y_0\right)\right]^2$

1. If D < 0 Then f(x,y) has a saddle point at $\left(x_0, y_0\right)$.

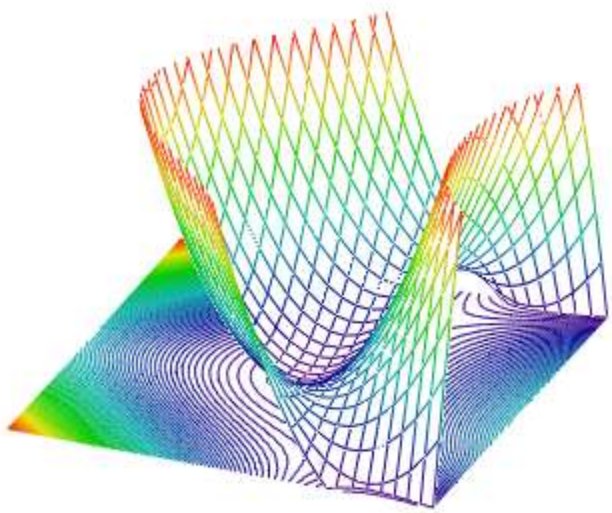
2. If D > 0 Then:

a. If $\frac{\delta^2 f}{\delta x^2}(x_0, y_0) > 0$ Then f(x,y) has a local minimum at (x_0, y_0)

b. If $\frac{\delta^2 f}{\delta x^2}(x_0, y_0) < 0$ Then f(x,y) has a local maximum at (x_0, y_0)

3. If D = 0 then No conclusion results.

Example 1 Let $f(x,y) := x^2 + 2\cdot y^2 - x^2\cdot y$ on the square [-3,3]x[-3,3]



$$\nabla f = (2x - 2xy)\cdot\vec{i} + \left(4y - x^2\right)\vec{j}$$

$x - xy = 0$

$4y - x^2 = 0$

In the first equation x = 0 or y = 1

If x = 0 then equation 2 yields y = 0 so we get the single critical point (0,0).

If y = 1 then equation 2 becomes $4 - x^2 = 0$ which yields x = $\pm$2. We get the 2 critical points

(2,1) and (-2,1).

We have the 3 critical points (0,0), (2,1) and (-2,1).

$$\frac{\delta^2 f}{\delta x^2} = (2 - 2y) \qquad \frac{\delta^2 f}{\delta y^2} = 4 \qquad \frac{\delta^2 f}{\delta y \delta x} = (-2 \cdot x)$$

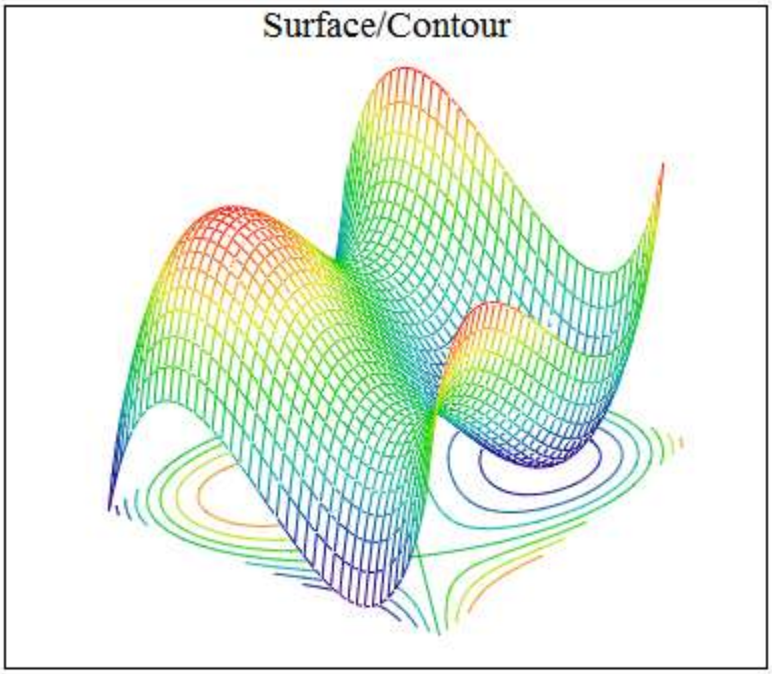
$$D = (2 - 2y)\cdot 4 - 4\cdot x^2$$

At (0,0) D = 8 and $\frac{\delta^2 f}{\delta x^2}$ = 2 Therefore there is a minimum at (0,0).

At (2,1) D = - 4 therefore there is a saddle at (2,1)

At (-2,1) D = -4 therefore there is a saddle at (-2,1).

Example 2 Let $f(x,y) := x^3 + y^3 - 3 \cdot x - 3 \cdot y$



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$\nabla f = (3 \cdot x^2 - 3) \cdot \vec{i} + (3y^2 - 3)\vec{j}$

$3 \cdot x^2 - 3 = 0$

$3y^2 - 3 = 0$

In the first equation x = ± 1 In the second equation y = ±1.

Therefore we have the 4 critical points (1,1),(-1,1),(1,-1) and (-1,-1)

$\frac{\delta^2 f}{\delta x^2} = 6x$ $\frac{\delta^2 f}{\delta y^2} = 6y$ $\frac{\delta^2 f}{\delta y \delta x} = 0$

$D = 36 \cdot xy$

At (1,1) D = 36 and $\frac{\delta^2 f}{\delta x^2}$ = 6 Therefore there is a minimum at (1,1).

At (1,-1) D = - 36 therefore there is a saddle at (1,-1)

At (-1,1) D = - 36 therefore there is a saddle at (-1,1).

At (-1,-1) D = 36 and $\frac{\delta^2 f}{\delta x^2}$ = -6 therefore there is a saddle at (-1,-1).

Why does the second partials test work ?

I won't formally prove it but try to give you an intuitive idea of why it works.

$$D = \frac{\delta^2 f}{\delta x^2}(x_0, y_0) \cdot \left[\frac{\delta^2 f}{\delta y^2}(x_0, y_0)\right] - \left[\frac{\delta^2 f}{\delta y \delta x}(x_0, y_0)\right]^2$$

If D > 0 then both $\frac{\delta^2 f}{\delta x^2}(x_0, y_0)$ and $\frac{\delta^2 f}{\delta y^2}(x_0, y_0)$ are both positive or both negative

If $\frac{\delta^2 f}{\delta x^2}(x_0, y_0)$ >0 then on the curve of intersection of f(x,y) and y = y_0 is concave up. But $\frac{\delta^2 f}{\delta y^2}(x_0, y_0)$ >0 and on the curve of intersection of f(x,y) and x = x_0 is concave up.

Locally the critical point is on the bottom of a bowl i.e. it is a minimum.

Similarly if both are negative we locally both curves of intersection are concave down and we are at the top of a bowl i.e. the critical point is a local maximum.

If D < 0 then one of the 2d partials is positive and the other is negative. If we approach the critical point along one of the curves of intersection it is concave up and if we approach along the other it is concave down. In other words a saddle.

See the graph below