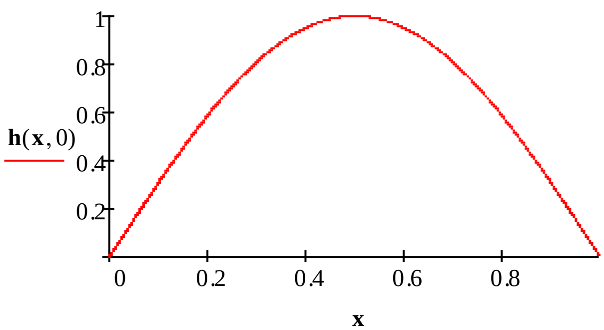
Second Partial Derivatives

***We will use δ   to denote partial derivatives***

Suppose we have a string fixed at x = 0 and x = 1 .

Let $\mathbf{h}(\mathbf{x},\mathbf{t}) := \cos(\mathbf{t})\cdot\sin(\pi\,\mathbf{x})$ be the height of the string at pt x at time t .

Below is the string a t = 0.




See the Animation String Vibration for the motion of the string

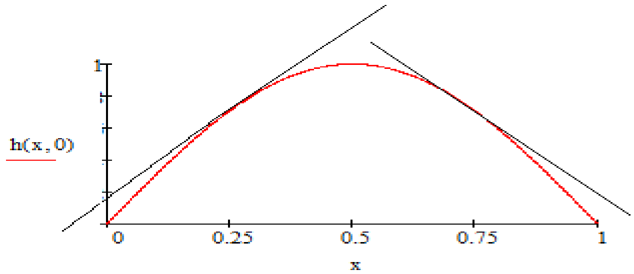
See also the Animation - String in 3-d    which shows the motion if we could see in time as well as space.

We have discussed the partial derivatives and will interpret them in terms of this vibrating string.

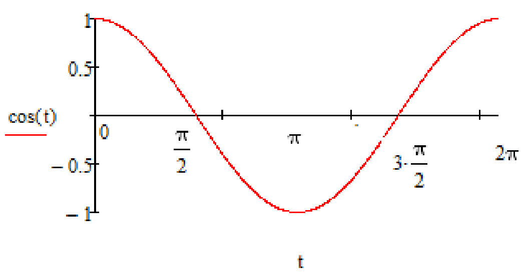
We will also introduce and discuss the second partial derivatives.

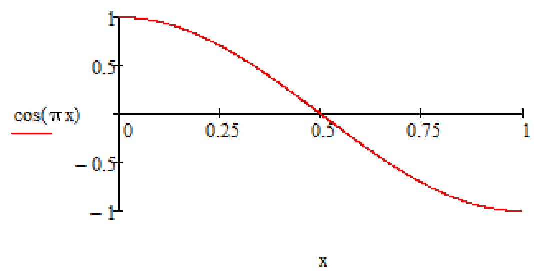
*1.* $\frac{\delta \mathbf{h}}{\delta \mathbf{x}}$

For each fixed time $\mathbf{t}_0$   $\mathbf{h} = \mathbf{h}(\mathbf{x}, \mathbf{t}_0)$ is a function of a single variable x. Therefore $\frac{\delta \mathbf{h}}{\delta \mathbf{x}}$ ***is simply the slope of the tangent line at each (x).*** Below we see the tangent lines at x = 1/4 and 3/4.




$\frac{\delta h}{\delta x} = \pi \cos(t_0) \cdot \cos(\pi x)$ Note $\frac{\delta h}{\delta x} > 0$ when both have the same sign and $\frac{\delta h}{\delta x} < 0$ when they have different signs




$0 \le t \le \pi/2$

Note that $\frac{\delta h}{\delta x} > 0$ if $0 \le x \le 1/2$ and the slope of the tangent line is positive. $\frac{\delta h}{\delta x} < 0$ if $1/2 \le x \le 1$ and the slope of the tangent line is negative.

For $\pi/2 \le t \le 3\pi/2$ this reverses.
For $3\pi/2 \le t \le 2\pi$ the situation is the same as for $0 \le t \le \pi/2$.

See the Animation Vibrating String with Tangent Line

2. $\frac{\delta^2 h}{\delta t \cdot \delta x}$

As the animation shows at each fixed x the tangent line changes in time. This is precisely what me mean

by $\frac{\delta}{\delta t}\left(\frac{\delta h}{\delta x}\right) = \frac{\delta^2 h}{\delta t \cdot \delta x}(x, t)$ -- the rate at which the slope of the tangent line at each x changes This is one of four second partial derivative and is one of two mixed partial derivatives.

$$\frac{\delta^2 h}{\delta t \cdot \delta x} = -\pi \sin(t) \cdot \cos(\pi x)$$

If we focus in on x = .25

$0 < t < \pi$ $\frac{\delta^2 h}{\delta t \delta x} < 0$ so the slope of the tangent line is decreasing

$\pi < t < 2\pi$ $\frac{\delta^2 h}{\delta t \delta x} > 0$ so the slope of the tangent line is increasing

You may want to view the animation Vibrating String with Tangent Line again .

Note the order of differentiation is right to left. If we use our subscript notation for partial differentiation

we write $f_{xt}$ and the order of differentiation is read left to right.

(The order of differentiation actually turns out not to matter if f and its first and second mixed partial derivatives are all continuous.)

3. $\frac{\delta^2 \mathbf{h}}{\delta \mathbf{x}^2}$

Recall $\frac{\delta \mathbf{h}}{\delta \mathbf{x}}$ is the slope of the tangent line at each x for fixed time $\mathbf{t}_0$. Then $\frac{\delta}{\delta \mathbf{x}}\left(\frac{\delta \mathbf{h}}{\delta \mathbf{x}}\right) = \frac{\delta^2 \mathbf{h}}{\delta \mathbf{x}^2}$

is the concavity of the string. $\frac{\delta^2 \mathbf{h}}{\delta \mathbf{x}^2} = -\pi^2 \cdot \cos(\mathbf{t}) \cdot \sin(\pi\, \mathbf{x})$. Note $\sin(\pi\, \mathbf{x}) > 0$ for all x so the sign of $\frac{\delta^2 \mathbf{h}}{\delta \mathbf{x}^2}$

is determined by the term $-\pi^2 \cdot \cos(\mathbf{t})$

$0 < t < \pi/2$ $\frac{\delta^2 \mathbf{h}}{\delta \mathbf{x}^2} < 0$ and at every pt (except the endpoints) The shape of the string is concave down.

$\pi/2 < t < 3\pi/2$ $\frac{\delta^2 \mathbf{h}}{\delta \mathbf{x}^2} > 0$ and at every pt (except the endpoints) The shape of the string is concave up.

$3\pi/2 < t < 2\pi$ $\frac{\delta^2 \mathbf{h}}{\delta \mathbf{x}^2} < 0$ and at every pt (except the endpoints) The shape of the string is concave down.

Again See the Animation Vibrating String with Tangent Line

4. $\frac{\delta \mathbf{h}}{\delta \mathbf{t}}$

This time we fix $x = x_0$ so $\frac{\delta \mathbf{h}}{\delta \mathbf{t}}$ is the velocity of the pt on the string $h(x_0,t)$ .

$$\frac{\delta \mathbf{h}}{\delta \mathbf{t}} = -\sin(\mathbf{t}) \cdot \sin(\pi\, \mathbf{x})$$

$\frac{\delta \mathbf{h}}{\delta \mathbf{t}} < 0$ for $0 < t < \pi$ the particle is moving down

$\frac{\delta \mathbf{h}}{\delta \mathbf{t}} > 0$ for $\pi < t < 2\pi$ the particle is moving up

Note for a single particle on the string the motion is analogous to the motion of a mass on a spring

See the Animation String Velocity.

5. $\frac{\delta^2 h}{\delta t^2}$

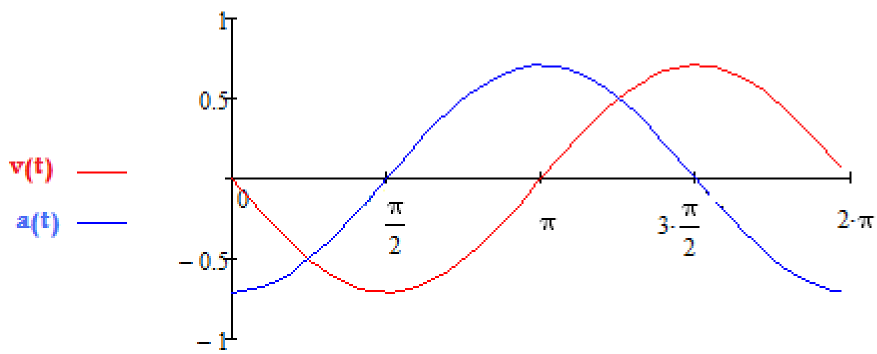
Simply $\frac{\delta^2 h}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta h}{dt}\right)$ is the rate of change of the velocity of a single particle i.e. the acceleration.

$$\frac{\delta^2 h}{\delta t^2} = -\cos(t)\cdot\sin(\pi\, x)$$

Recall from one dimensional motion that a particle speeds up when the velocity and acceleration have the same sign and the particle is slowing down when the velocity and acceleration have opposite signs.

The velocity is $\frac{\delta h}{\delta t} = -\sin(t)\cdot\sin(\pi\, x)$ and the acceleration is $\frac{\delta^2 h}{\delta t^2} = -\cos(t)\cdot\sin(\pi\, x)$

In the animation we focus on x = .25, however what is true at x = .25 is true for all x.



$0 < t < \pi/2$ $\frac{\delta^2 h}{\delta t^2}$ and $\frac{\delta h}{\delta t}$ are both negative so the particle is moving down and speeds up

$\pi/2 < t < \pi$ $\frac{\delta^2 h}{\delta t^2} > 0$ and $\frac{\delta h}{\delta t} < 0$ so the particle slows down and comes to a stop at $t = \pi$ .

$\pi < t < 3\pi/2$ $\frac{\delta^2 h}{\delta t^2}$ and $\frac{\delta h}{\delta t}$ are both positive so the particle starts moving up picking up speed

$3\pi/2 < t < 2\pi$ $\frac{\delta^2 h}{\delta t^2} < 0$ and $\frac{\delta h}{\delta t} > 0$ so the particle slows down coming to a stop at $t = 2\pi$ .

Again this is analogous to the motion of a mass on a spring.

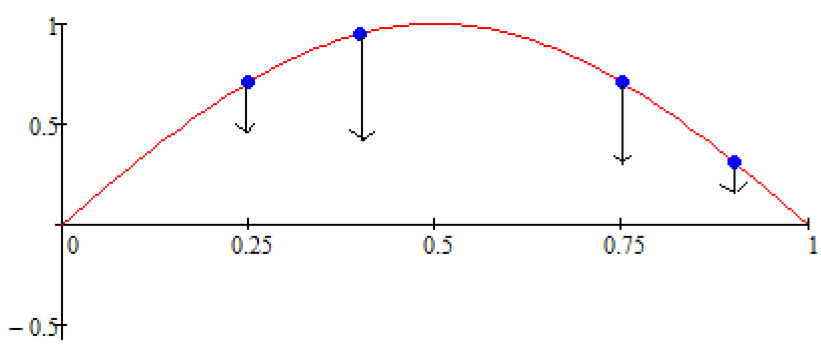
You may want to again view the Animation String Velocity.

6. $\frac{\delta^2 h}{\delta x \delta t}$

By $\frac{\delta^2 h}{\delta x \delta t}$ we mean $\frac{\delta}{\delta x}\left(\frac{\delta h}{dt}\right)$ .

Now $\frac{\delta h}{dt}$ is the velocity of a particle of the string. It follows $\frac{\delta}{\delta x}\left(\frac{\delta h}{dt}\right)$ is the rate at which the velocities of the particles change as we move from left to right along the string.

$$\frac{\delta^2 h}{\delta x \delta t} = -\pi \sin(t) \cdot \cos(\pi x)$$



Consider what happens at t = 0. If $0 < x < .5$ then $\frac{\delta^2 h}{\delta x \delta t} < 0$ meaning $\frac{\delta h}{\delta t}$ is decreasing as we move to the right along the string. For example as we move from x = .25 to x = .4 $\frac{\delta^2 h}{\delta x \delta t} < 0$ meaning the velocity is

decreasing i.e. is more negative-- meaning a particle to the right of x is traveling downward faster which is necessary for the string to maintain its shape.

For $.5 < x < 1$ $\frac{\delta^2 h}{\delta x \delta t} > 0$ meaning $\frac{\delta h}{\delta t}$ is increasing as we move to the right along this string .For example

At x = .75 cos(π .75) < 0 so $\frac{\delta^2 h}{\delta x \delta t} > 0$ meaning as we move to from .75 to .9 the velocity is less negative

i.e. is increasing which means a particle to the right of x is moving slower.

See the animation - String Velocities.

Example Let $f(x, y) = x^3 \cdot \sin(y)$

Compute all second derivatives

$f_x = 3 \cdot x^2 \sin(y)$

$f_y = x^3 \cos(y)$

$f_{xy} = 3 \cdot x^2 \cos(y)$

$f_{yx} = 3 \cdot x^2 \cos(y)$

$f_{xx} = 6 \cdot x \sin(y)$

$f_{yy} = -x^3 \sin(y)$