One-Dimensional Motion with and without air resistance.

One of the first applications you ever considered in Calculus 1 was the motion of an object in free fall.

The equations of motion are : position $y(t) = -16t^2 + v_0 t + y_0$

velocity $v(t) = -32t + v_0$

acceleration $a(t) = -32$

We are now going to consider the case where we add in air resistance.

We'll consider the case in which the air resistance force is proportional to the velocity.

Starting with Newton's Second Law of Motion : $m\cdot\frac{dv}{dt} = -mg - \beta v$. Where β is called the drag coefficient and depends on the physical properties of the body.

$$\frac{dv}{dt} = -g - \frac{\beta}{m}\cdot v.$$

We can write this in the form the first order linear DE:

$$\frac{dv}{dt} + \frac{\beta}{m}\cdot v = -g$$

The integrating factor is : $e^{\left(\int \frac{\beta}{m} dt\right)}$

We obtain

$$e^{\left(\int \frac{\beta}{m} dt\right)} = e^{\left(\frac{\beta}{m}\cdot t\right)}$$

$$\frac{d\left[\left[e^{\left(\frac{\beta}{m}\cdot t\right)}\cdot v\right]\right]}{dt} = -g\cdot e^{\left(\frac{\beta}{m}\cdot t\right)}$$

Then :

$$e^{\left(\frac{\beta}{m}\cdot t\right)}\cdot v = \int -g\cdot e^{\left(\frac{\beta}{m}\cdot t\right)} dt = \frac{-m\cdot g}{\beta}\cdot e^{\left(\frac{\beta}{m}\cdot t\right)} + c$$

$$v(t) = \frac{-m \cdot g}{\beta} + c \cdot e^{-\left(\frac{\beta}{m} \cdot t\right)}$$

Applying the initial condition v(0) = v_0 we have finally $v = \frac{-mg}{\beta} + \left(m \cdot \frac{g}{\beta} + v_0\right) \cdot e^{\frac{-\beta t}{m}}$

Recall in the English system the weight of an object is mg so to formulate this in terms weight and not mass we would have $v = -\left(\frac{w}{\beta}\right) + \left(\frac{w}{\beta} + v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}}$

Let's Compare the velocity in the cases with and without air resistance.

Let v_f represent the velocity of an object in free fall and v_r represent the velocity of an object with air resistance.

$g := 32$

$v_0 := 64$

$\beta := 5$

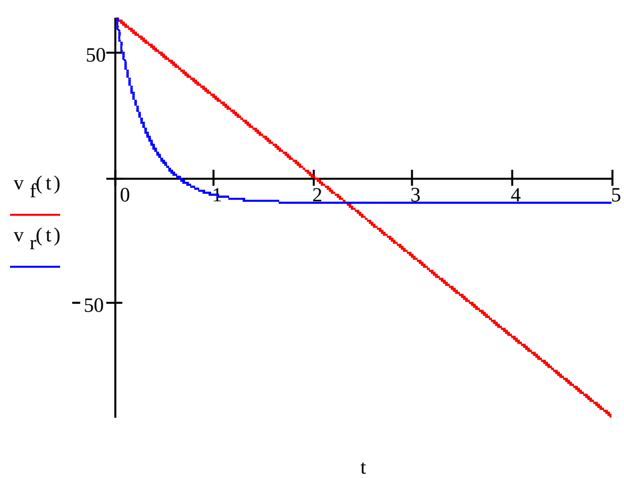
$m := 1.563$

$w := 50$

$y_0 := 80$

$v_f(t) := -g \cdot t + v_0$

$$v_r(t) := -\left(\frac{w}{\beta}\right) + \left(\frac{w}{\beta} + v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}}$$



Notice with air resistance the velocity appears to approach a limiting value whereas in the free fall case the velocity continues to decrease. This limiting value is called the terminal velocity and we can see

$$\lim_{t \to \infty} -\left(\frac{w}{\beta}\right) + \left(\frac{w}{\beta} + v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}} = \frac{-w}{\beta}.$$

$g := 32$

$v_0 := 64$

$m := 1.563$

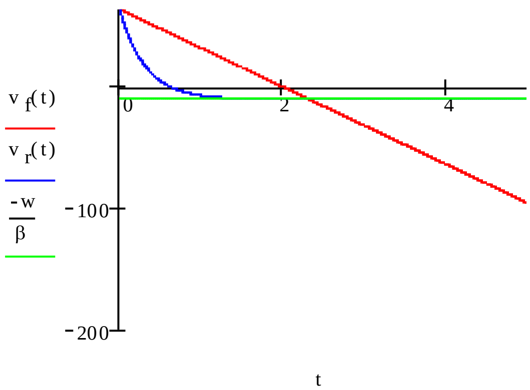
$w := 50$

$y_0 := 80$

$\beta := 5$

$v_f(t) := -g \cdot t + v_0$

$$v_r(t) := -\left(\frac{w}{\beta}\right) + \left(\frac{w}{\beta} + v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}}$$



View animation VELOCITY and note as β decreases vf approaches vr and the terminal velocity –> -∞

To prove this formally show using some algebraic manipulation and L'Hoptital's Rule

$$\lim_{\beta \to 0} \frac{-mg}{\beta} + \left(m \cdot \frac{g}{\beta} + v_0\right) \cdot e^{\frac{-\beta t}{m}} = -g \cdot t + v_0$$

Let’s Consider The position functions. Using the same conventions as before we have:

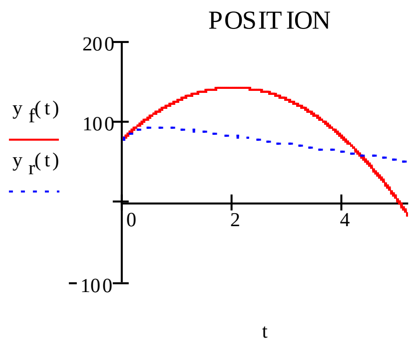
$y_f(t) := -16 \cdot t^2 + v_0 \cdot t + y_0.$

Using $\frac{dy_r}{dt} = v_r(t) = -\left(\frac{w}{\beta}\right) + \left(\frac{w}{\beta} + v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}}$ we can separate variables and integrate to obtain:

$$y_r(t) := \left(-\frac{w}{\beta} \cdot t\right) - \frac{w}{\beta \cdot g} \cdot \left[\left(\frac{w}{\beta} + v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}}\right] + C.$$

Applying the initial condition we obtain $C = y_0 + \frac{w}{\beta \cdot g} \cdot \left(\frac{w}{\beta} + v_0\right)$.

Finally we obtain: $y_r(t) := \left[-\frac{w}{\beta}\cdot t - \frac{w}{\beta\cdot g}\cdot\left(\frac{w}{\beta}+v_0\right)\cdot e^{\frac{-\beta\cdot g\cdot t}{w}}\right] + y_0 + \frac{w}{\beta\cdot g}\cdot\left(\frac{w}{\beta}+v_0\right).$



View animation POSITION and Again as β decreases and you will see the position function for the air resistance case approach the position function for the case with no air resistance

Let’s consider the two together

$g := 32$

$v_0 := 64$

$\beta := 5$

$m := 1.563$

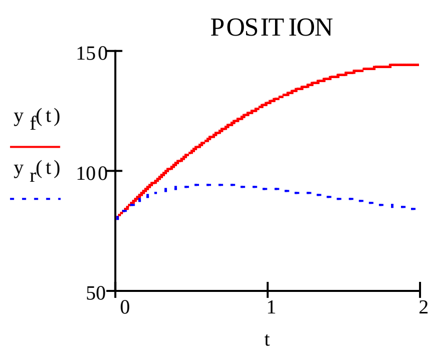
$w := 50$

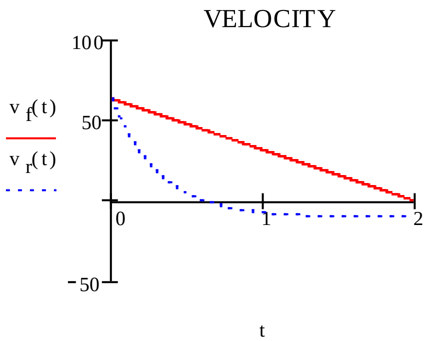
$y_0 := 80$

$y_f(t) := -16\cdot t^2 + v_0\cdot t + y_0$

$$y_r(t) := \left[-\frac{w}{\beta}\cdot t - \frac{w}{\beta\cdot g}\cdot\left(\frac{w}{\beta}+v_0\right)\cdot e^{\frac{-\beta\cdot g\cdot t}{w}}\right] + y_0 + \frac{w}{\beta\cdot g}\cdot\left(\frac{w}{\beta}+v_0\right)$$

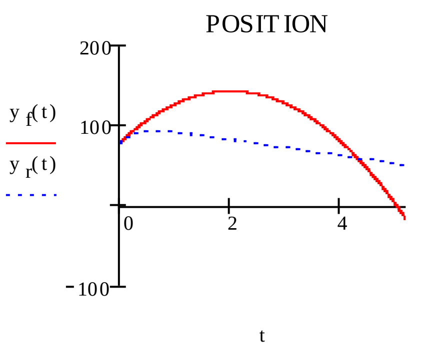
In the first graph let's consider what happens in the first 2 seconds Because of air resistance y_r reaches its maximum height and starts falling before y_f.

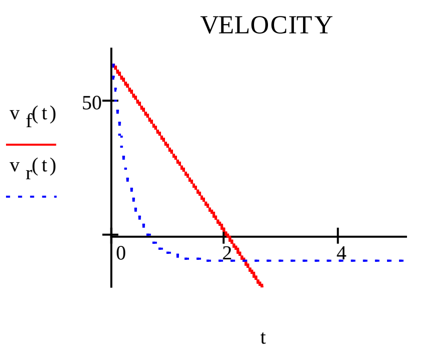




[View animation FIRST 2 SECS]

In this second graph We consider the entire trajectories. Note that without air resistance the particle rises more quickly then as it starts to fall it speeds up catches and passes the particle with air resistance.
The reason being of course that with air resistance the particle reaches its terminal velocity. Note also that as the particle reaches terminal velocity the plot of v_r approaches a constant value and the position function is becoming linear.





[View animation ENTIRE TRAJECTORY]

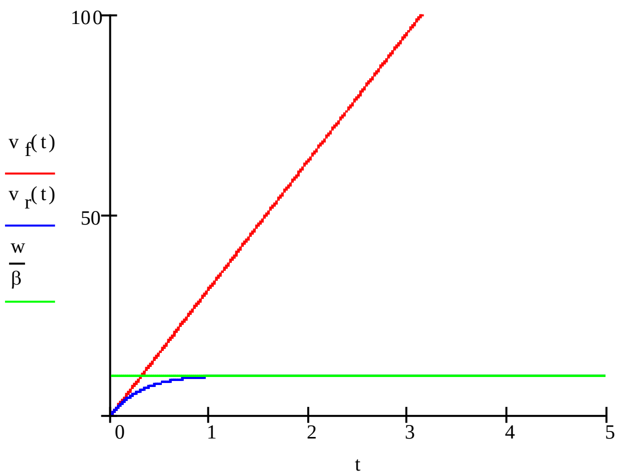
In certain problems such as releasing an object or parachute problems since the motion is one direction we are more concerned with speed and distance rather than position and velocity. In these cases we take down as the positive direction.

The DE takes the form $m \cdot \frac{dv}{dt} = mg - \beta v$

and the solution is $v = \frac{mg}{\beta} - \left(\frac{mg}{\beta} - v_0\right) \cdot e^{\frac{-\beta t}{m}}$.   (note :Typically in these cases v_0 is 0.)

$g := 32$
$v_0 := 0$
$\beta := 5$
$m := 1.563$
$w := 50$
$y_0 := 80$
$v_f(t) := g \cdot t + v_0$

$$v_r(t) := \frac{w}{\beta} - \left(\frac{w}{\beta} - v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}}$$



For distance we then have:
 Using the same conventions as before we have:

$$y_f(t) := 16 \cdot t^2 + v_0 \cdot t + y_0.$$

Using $\frac{dy_r}{dt} = v_r(t) = \frac{w}{\beta} - \left(\frac{w}{\beta} - v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}}$ we can separate variables and integrate to obtain:

$$y_r(t) := \frac{w}{\beta} \cdot t + \frac{w}{\beta \cdot g} \cdot \left(\frac{w}{\beta} - v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}} + C.$$

Applying the initial condition we obtain $C = y_0 - \frac{w}{\beta \cdot g} \cdot \left(\frac{w}{\beta} - v_0\right)$.

Finally we obtain: $y_r(t) := \frac{w}{\beta}\cdot t + \frac{w}{\beta\cdot g}\cdot\left(\frac{w}{\beta} - v_0\right)\cdot e^{\frac{-\beta\cdot g\cdot t}{w}} + y_0 - \frac{w}{\beta\cdot g}\cdot\left(\frac{w}{\beta} - v_0\right)$.

$g := 32$

$v_0 := 0$

$\beta := 5$

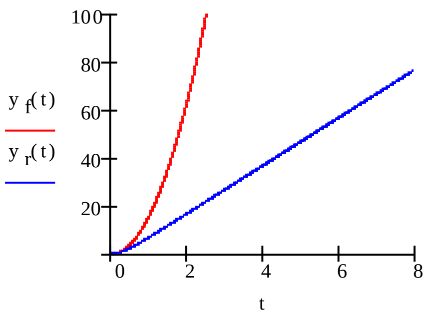
$m := 1.563$

$w := 50$

$y_0 := 0$

$y_f(t) := 16\cdot t^2 + v_0\cdot t + y_0$

$$y_r(t) := \frac{w}{\beta}\cdot t + \frac{w}{\beta\cdot g}\cdot\left(\frac{w}{\beta} - v_0\right)\cdot e^{\frac{-\beta\cdot g\cdot t}{w}} + y_0 - \frac{w}{\beta\cdot g}\cdot\left(\frac{w}{\beta} - v_0\right)$$



Here we have taken the release point y(0) to be 0.

Note in 2 secs without air resistance the object has already fallen 100 ft whereas with air resistance the object has only fallen about 20 ft.

Summary

Velocity

Metric System

$$m\cdot\frac{dv}{dt} = -mg - \beta v$$

$$v = \frac{-mg}{\beta} + \left(m\cdot\frac{g}{\beta} + v_0\right)\cdot e^{\frac{-\beta t}{m}}$$

English System

$$\frac{dv}{dt} = -g - \frac{\beta g}{w} \cdot v$$

$$v = -\left(\frac{w}{\beta}\right) + \left(\frac{w}{\beta} + v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}}$$

Speed

$$m \cdot \frac{dv}{dt} = mg - \beta v$$

$$v = \frac{mg}{\beta} - \left(\frac{mg}{\beta} - v_0\right) \cdot e^{\frac{-\beta t}{m}}.$$

$$\frac{dv}{dt} = g - \beta \cdot \frac{g}{w} \cdot v$$

$$v = \frac{w}{\beta} - \left(\frac{w}{\beta} - v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}}$$

Position

Metric System

$$y_r(t) := -\frac{m \cdot g}{\beta} \cdot t + \frac{m}{\beta} \cdot \left(\frac{-m \cdot g}{\beta} - v_0\right) \cdot e^{\frac{-\beta \cdot t}{m}} + y_0 - \frac{m}{\beta} \cdot \left(\frac{-mg}{\beta} - v_0\right)$$

English System

$$y_r(t) := -\frac{w}{\beta} \cdot t + \frac{w}{\beta \cdot g} \cdot \left(\frac{-w}{\beta} - v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}} + y_0 - \frac{w}{\beta \cdot g} \cdot \left(\frac{-w}{\beta} - v_0\right)$$

Distance

Metric System

$$y_r(t) := \frac{mg}{\beta} \cdot t + \frac{m}{\beta} \cdot \left(\frac{mg}{\beta} - v_0\right) \cdot e^{\frac{-\beta \cdot t}{m}} + y_0 - \frac{m}{\beta} \cdot \left(\frac{mg}{\beta} - v_0\right)$$

English System

$$y_r(t) := \frac{w}{\beta} \cdot t + \frac{w}{\beta \cdot g} \cdot \left(\frac{w}{\beta} - v_0\right) \cdot e^{\frac{-\beta \cdot g \cdot t}{w}} + y_0 - \frac{w}{\beta \cdot g} \cdot \left(\frac{w}{\beta} - v_0\right)$$